August 2023

E & M TEST BANK

1. Charge in an Atom [500 level]
   A certain charge distribution $\rho(\mathbf{r})$ generates an electric potential $\Phi(\mathbf{r}) = \frac{e_0}{a} e^{-2r/a}\left(\frac{a}{r}+1\right)$, with constants e_0 and a.
   (a) Find the electric field $\mathbf{E}(\mathbf{r})$.
   (b) Find the charge distribution $\rho(\mathbf{r})$.

2. Charged Sphere [500 level]
   An insulating sphere of radius R carries a surface charge $\sigma(\theta) = k\cos\theta$, where θ is the polar angle and k is a constant
   (a) Find the dipole moment $\vec{p}$ of the sphere.
   (b) Find the potential outside the sphere.
   (c) Find the potential inside the sphere.

3. Electrostatics [500 level]
   A conducting sphere of radius a, at potential V_0, is surrounded by a thin concentric insulating shell of radius b, on which is glued a surface charge $\sigma = k\cos\theta$, where k is a constant and θ is the polar angle.
   (a) Find the potential for $r > b$.
   (b) Find the potential for $a < r < b$.
   (c) Find the induced surface charge on the conductor.
   (d) Find the total charge of the system.

4. Electrostatics [500 level]
   A point dipole $\vec{p} = p\hat{z}$ is located at $r = 0$. A thick, grounded spherical conducting shell surrounds the dipole between radii a and b.
   (a) Find the potential inside the conductor ($r < a$).
   (b) Find the total induced charge on the inner surface of the conductor.
   (c) Find the potential outside the conductor ($r > b$).

5. Electrostatics [500 level]
   The potential on a sphere of radius a is $k\cos 3\theta$, where k is some constant.
   (a) Find the potential inside the sphere.
   (b) Find the potential outside the sphere.
   (c) Find the surface charge on the sphere.

6. Electrostatics [500 level]
   Two grounded spherical shells are located at radii a and c. Between them at

radius b (with $a < b < c$) is another concentric shell carrying a uniform surface charge σ.
(a) Express the electric field $\vec{E}$ in the $a < r < b$ and $b < r < c$ regions in terms of a, b, c, σ, and the unknown charge q_a contained on the inner conductor.
(b) What is the total charge of the system?
(c) Determine the potential and use it to find q_a.

7. Method of Images [700 level]
A grounded conducting plane is located at $z = 0$ and an electric dipole $\vec{p} = p\hat{z}$ is fixed at $(0, 0, d)$.
(a) What configuration of image sources will satisfy the boundary conditions in this problem?
(b) Find the electric potential above the conductor and far away ($r \gg d$) from the dipole.
(c) What is the total charge induced on the conducting plane?

8. Charge and Conductor [700 level]
A charge q is located a distance d away from the center of a planar conducting disk of radius $R \gg d$. (This charge is situated on the axis of the disk.) We will find the surface charge density on the disk; only the lowest order terms in the small parameter d/R will be required.
(a) Find an approximate image charge solution, as if the disk were grounded, and determine the (approximate) surface charge density in that case.
(b) What is the (approximate) total charge on the grounded disk?
(c) The isolated disk is actually overall charge neutral. How is the remaining charge (not accounted for by the image charge problem) distributed over its surface? It may help to know that the surface charge density on an isolated conducting disk of radius a and charge Q is

$$\sigma = \frac{Q}{2\pi a\sqrt{a^2 - x^2 - y^2}}.$$

9. Electrostatics [500 level]
The two hemispheres of a hollow conducting spherical shell of radius a are separated by a thin insulating ring. One hemisphere is maintained at a potential $+V_0$ and other at potential $-V_0$.
(a) Find the potential at the center of the sphere.
(b) Find the force on a small charge q placed at the center of the sphere.
(c) Why was it necessary to consider a *small* charge in part (b)?

10. Capacitance [500 level]
A capacitor is made up of three coaxial cylindrical conducting surfaces, at

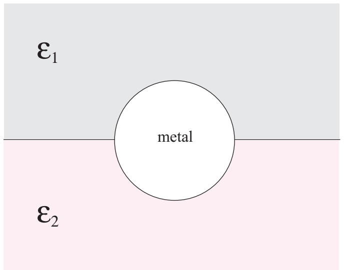
radii a, b, and c (in increasing order). The two inner surfaces (radii a and b) are wired together, forming one terminal of the capacitor. The surface at radius c forms the other terminal.
(a) Find the electric field in the regions between radii a and c.
(b) Find the capacitance per unit length of this configuration.
(c) Find the energy stored per unit length, if the charge per unit length stored on the capacitor is Λ.

11. Energy Balance [500 level]
A conductive soap bubble carries a charge Q and has surface tension (energy per unit surface area) Σ.
(a) Calculate the electrostatic energy as a function of the bubble radius.
(b) What condition determines the equilibrium radius of the bubble?
(c) Find the equilibrium radius of the bubble.

12. Electrostatic Energy [500 level]
A gas of charged particles, with total charge Q, is confined within a spherical container of radius a.
(a) How is the charge of the gas distributed at very high temperatures?
(b) Find the electrostatic energy in the high-temperature configuration.
(c) How is the charge distributed at very low temperatures?
(d) Find the potential energy in this low-temperature configuration.

13. Dielectrics [500 level]
A dielectric sphere of radius a with a dielectric constant ϵ is placed in an initially uniform electric field which at large distances from the sphere is directed along the z axis and has magnitude E_0. Both inside and outside the sphere there are no free charges. Recall that the electric displacement field $\mathbf{D}$ is defined by the formula $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E}$ connecting it with the electric field $\mathbf{E}$ and the polarization vector $\mathbf{P}$.
(a) Find the electric potential $\Phi(r, \theta)$, both inside and outside the sphere. Keep in mind that the problem has axial symmetry and the boundary conditions for an interface of dielectrics are such that the tangential components of the electric field and the normal component of the displacement field are continuous at $r = a$.
(b) Compute the electric field $\mathbf{E} = -\nabla \Phi$.
(c) What is the bound surface charge density σ_b?

14. Dielectrics [500 level]
A conducting sphere of radius a carrying a charge Q floats exactly half-submerged in a non-conducting liquid of permittivity ϵ_2. The space above the liquid has permittivity ϵ_1.



(a) State the uniqueness theorem for the electrostatic potential.
(b) Find the electric field outside the sphere.
(c) Find the charge density on the surface of the sphere.

15. Dielectric [500 level]
A point charge q is located a distance $r \gg a$ away from a dielectric sphere of radius a.
(a) Is the force between the charge and the sphere attractive or repulsive?
(b) If the charge q is increased to $2q$, how must the distance r be changed so that the force remains the same?

16. Dielectric [700 level]
A sphere of radius a, made of linear dielectric material with permitivity ϵ, has embedded in it a uniform free charge density ρ.
(a) Find the electric displacement $\mathbf{D}$.
(b) Find the electric field $\mathbf{E}$, inside and outside of the sphere.
(c) Find the potential at the center of the sphere.

17. Dielectric Cube [500 level]
A dielectric cube of side a, centered at the origin, contains a "frozen-in" polarization $\mathbf{P} = k\mathbf{r}$, where k is a constant.
(a) Find the bound surface charges σ_b.
(b) Find the bound volume charge ρ_b.
(c) Show that the total bound charge is zero.

18. Dielectric Capacitor [500 level]
A coaxial cable consists of a copper wire of radius a, surrounded by a concentric copper tube of radius c. Part of the space in between (from radius b to c) is filled with a linear dielectric material with dielectric constant κ. The inner copper wire carries a charge per unit length λ, and the tube carries $-\lambda$.
(a) Find the electric displacement $\mathbf{D}$.
(b) Find the electric field $\mathbf{E}$.
(c) Find the capacitance per unit length of this cable.

19. Dipole [500 level]
A physical dipole consists of a charge q located at $(0, 0, d/2)$ and another charge

$-q$ located at $(0, 0, -d/2)$.
(a) Write down the exact potential for this system of charges.
(b) What is the dipole moment?
(c) Expand the exact potential to $\mathcal{O}[(d/r)^3]$ to find the quadrupole and octopole contributions to the potential in the xy-plane.

20. Dipoles [500 level]
Two electric dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ of equal magnitude p interact through their electric fields. They are fixed at locations 0 and $\mathbf{r} = (x, y, z)$, respectively, but are free to rotate.
(a) Show that the following two configurations are in equilibrium:
(i) Both dipoles point along $\mathbf{r}$ and are parallel to one another.
(ii) Both dipoles point perpendicular to $\mathbf{r}$ and are antiparallel.
(b) Determine whether the two configurations considered in part (a) are stable.

21. Multipoles [500 level]
A sphere of radius a, centered at the origin, carries a charge density $\rho(r, \theta) = k\frac{a}{r^2}(a - 2r)\sin\theta$, where k is a constant.
(a) Find the total charge of the sphere.
(b) Find the dipole moment of the sphere.
(c) Find the approximate potential far away from the sphere.

22. Multipoles [700 level]
A thin insulating rod, running from $z = -a$ to $z = a$ carries a line charge $\lambda(z)$. For each of the following expressions for λ, find the leading term in the multipole expansion of the potential.
(a) $\lambda = K\cos(\pi z/2a)$
(b) $\lambda = K\sin(\pi z/a)$
(c) $\lambda = K\cos(\pi z/a)$.

23. Poisson Equation [700 level]
A charge q is located at the center of a grounded, conducting cube of side a. You will determine the potential inside the cube using separation of variables.
(a) Show how to reduce the solution of this problem to the solution of a Dirichlet problem for the Laplace equation.
(b) Show how the Dirichlet problem can be further reduced to a simpler Dirichlet problem in which the boundary conditions involve a nonzero potential on only one face of the cube.
(c) Set up a solution of the simplified problem. You do not need to evaluate any integrals explicitly.
(d) What other technique could have been used to solve this problem?

24. Moving Charge [500 level]
A phonograph record carries a uniform charge density σ fixed to its surface. The record rotates at an angular velocity ω.
(a) What is the surface current density at a distance r from the center of the record?
(b) Find the electric field on the axis of the record.
(c) Find the magnetic field on the axis of the record.

25. Polygonal Wire [500 level]
(a) Find the magnetic field at the center of a square loop of wire, carrying a current I. Let a be the distance from the center to the midpoint of each side.
(b) Find the magnetic field at the center of a regular n-sided polygon, carrying a current I. Again, let a be the apothem distance.
(c) Show that as $n \to \infty$, you recover the result for a circular loop.

26. Magnetostatics [700 level]
A magnetic field $\mathbf{B}$ in a current-free region is cylindrically symmetric with components $B_z(\rho, z)$ and $B_\rho(\rho, z)$. The axial field $B_z(0, z)$ on the axis of symmetry is known and is a slowly varying function of z.
(a) Show that near the axis the radial and axial components of $\mathbf{B}$ are approximately

$$\begin{aligned} B_z(\rho, z) &= B_z(0, z) - \frac{\rho^2}{4}\left[\frac{\partial^2 B_z(0, z)}{\partial z^2}\right] \\ B_p(\rho, z) &= -\frac{\rho}{2}\left[\frac{\partial B_z(0, z)}{\partial z}\right] + \frac{\rho^3}{16}\left[\frac{\partial^3 B_z(0, z)}{\partial z^3}\right] \end{aligned}$$

(b) What is the relevant criterion defining being "near" the axis?

27. Convection Currents [500 level]
Suppose there are two infinite straight line charges λ, located at $x = 0$, $y = 0$ and $x = d$, $y = 0$. The line charges are moving, in parallel, with velocity $v\hat{\mathbf{z}}$.
(a) What is the electrostatic repulsion between the two lines of charge?
(b) What is the magnetic attraction between them?
(c) How fast would the charge have to be moving for the electric and magnetic forces to cancel?

28. Convection Currents [500 level]
A large parallel plate capacitor consists of two plates separated by a distance d, carrying uniform surface charges σ on the upper plate and $-\sigma$ on the lower plate. The plates are moving with equal (and constant) speeds v in the plane of the capacitor.
(a) Find the magnetic field between the plates.

(b) Find the magnetic field above and below the plates.
(c) Find the magnetic force per unit area on the upper plate, including its direction.
(d) At what speed would the magnetic and electric forces between the plates balance?

29. Vector Potential [500 level]
A finite wire of length a runs from $z = -a/2$ to $z = a/2$ and carries a current I.
(a) Find the magnetic field.
(b) Find the magnetic vector potential in the Coulomb gauge.
(c) Show that the $\nabla \times \mathbf{A} = \mathbf{B}$.

30. Higher Potentials [500 level]
Just as the fact that $\nabla \cdot \mathbf{B} = 0$ allows us to express $\mathbf{B}$ as the curl of a vector potential, the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ permits us to write $\mathbf{A}$ itself as a curl of a "higher" potential, $\nabla \times \mathbf{W} = \mathbf{A}$.
(a) Find a formula for $\mathbf{W}$ as an integral over $\mathbf{B}$ that is valid if $\mathbf{B}$ vanishes sufficiently rapidly at infinity.
(b) Determine $\mathbf{W}$ for a uniform magnetic field, for which $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$.
(c) Find $\mathbf{W}$ inside and outside an infinite circular solenoid.

31. Vector Potential [500 level]
A current is flowing in a long straight wire. The wire has a cylindrical cross section of radius a, and the current I is uniformly distributed across the interior of the wire.
(a) Find the vector potential $\mathbf{A}$ outside the wire. Specify the gauge you are using.
(b) Find the vector potential $\mathbf{A}$ inside the wire.
(c) What happens to $\mathbf{A}$ at the surface of the wire?

32. Vector Potential [700 level]
(a) Find a magnetic vector potential of a uniform surface current $\mathbf{K}$, both above and below the surface.
(b) Determine whether your $\mathbf{A}$ satisfies $\nabla \cdot \mathbf{A} = 0$. If it does not, find a gauge transformation so that the transformed $\mathbf{A}'$ satisfies the Coulomb gauge condition $\nabla \cdot \mathbf{A}' = 0$

33. Time-Dependent Magnetization [500 level]
A infinitely long cylinder of radius a contains a magnetization parallel to the axis. The magnetization $\mathbf{M} = kt\hat{\mathbf{z}}$ is spatially uniform inside the cylinder but varies in time.
(a) Find the magnetic field $\mathbf{B}(t)$.

(b) Find the electric field $\mathbf{E}(t)$ outside the cylinder.
(c) Find $\mathbf{E}$ inside the cylinder.

34. Ferromagnetism [500 level]
A ferromagnetic cube of side a (with edges oriented parallel to the x, y, and z axes) contains a constant magnetization $\mathbf{M} = M\hat{\mathbf{z}}$.
(a) Find the bound currents.
(b) Are the bound currents divergenceless?
(c) Find the magnetic field far away from the cube.

35. Ferromagnetism [500 level]
A permanent magnet in the shape of right circular cylinder of radius a has a uniform magnetization $\mathbf{M}$ pointing parallel to its axis.
(a) Determine $\mathbf{H}$ from Ampere's Law if the cylinder is infinite long.
(b) For the same infinite cylinder, determine $\mathbf{B}$ inside and outside the cylinder.
(c) Where do the arguments in parts (a) and (b) break down for a cylinder of finite length?

36. Magnetization [500 level]
An infinitely long cylinder of radius R carries a magnetization $\mathbf{M} = k\rho^2\hat{\phi}$ (in cylindrical coordinates where ρ is the distance from the axis of the cylinder and $\hat{\phi}$ is the unit vector in the circumferential direction), for some constant k.
(a) Find the field $\mathbf{B}$ due to $\mathbf{M}$ *inside* the cylinder.
(b) Find the field $\mathbf{H}$ inside the cylinder.
(c) Find the field $\mathbf{B}$ outside the cylinder.

37. Mutual Inductance [500 level]
A small circular solenoid is located inside a much larger solenoid. The large solenoid has radius R_1, length ℓ_1, and is wound with n_1 turns of wire per unit length. The small solenoid has radius R_2, length ℓ_2, and is wound with n_2 turns of wire per unit length. The two solenoids are not coaxial; the axis of the smaller one is oriented at a $45°$ angle to the axis of the large one.
(a) Find the flux threaded through the small solenoid, due to the field of the large solenoid. Take the large solenoid to be carrying a current I.
(b) Find the mutual inductance of the two solenoids.

38. Changing Current [500 level]
A square loop of wire, with side a and resistance R, lies in the xy-plane, with vertices at $(0, d, 0)$, $(a, d, 0)$, $(a, d + a, 0)$, and $(0, d + a, 0)$. A current-carrying wire runs along the x-axis, initially carrying a current I.
(a) Find the magnetic flux through the square loop.
(b) At time $t = 0$, the wire is cut, and the current drops to zero. How much total charge flows past a given point in the square loop as a result of the change

in **B**?
(c) Which direction does the charge flow?

39. Faraday's Law Toroid [500 level]
A thin toroidal coil of wire has a rectangular cross section, with inner radius a, outer radius $a + w$, and height d ($w \ll a$ and $d \ll a$). It carries a total of N tightly wound turns of wire, in which the current is slowly increasing at a constant rate: $dI/dt = k$.
(a) Find the instantaneous magnetic field profile inside the toroid.
(b) Find the magnetic field outside the toroid.
(c) Find the electric field at a point a distance z above the center of the toroid, using an analogy between Faraday's Law and Ampere's Law.

40. Time-Dependent Fields [500 level]
Suppose that a set of time-dependent electric and magnetic fields are

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt - r)\hat{\mathbf{r}} \\ \mathbf{B}(\mathbf{r}, t) &= 0, \end{aligned}$$

where θ is the unit step function.
(a) Show that these fields satisfy all of Maxwell's equations.
(b) Find the charge and current densities that generate these fields.
(c) What physical configuration gives rise to these fields?

41. Continuity Equation [500 level]
(a) Show that Ampere's Law (without the displacement current term), can only hold for divergenceless currents.
(b) Show how to derive the continuity equation for conservation of charge, using the full Maxwell's Equations.

42. Electrodynamics [500 level]
A charged, highly conductive sphere of radius a, initially at a potential V_0 relative to ground, is fixed inside an infinite block of material—a linear dielectric with dielectric constant κ and a small conductivity σ.
(a) What current density **J** is generated in the material? (You may assume σ is small enough that space charge effects can be neglected.)
(b) Find the displacement current and the magnetic field.
(c) Determine the potential on the sphere as a function of time.

43. Waveguide [500 level]
Consider an infinite grounded conducting metal pipe with rectangular cross section $a \times b$, with $a \geq b$

(a) Show that transverse electric (TE) waves can propagate through this waveguide.
(b) What is the smallest possible frequency for such waves?
(c) What are the phase and group velocities for the lowest TE modes?

44. Relativistic Electrodynamics [700 level]
A perfectly conducting sphere of radius a moves with velocity $\mathbf{v} = v\hat{\mathbf{z}}$ through a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{y}}$.
(a) What is the electric field $\mathbf{E}$ in the rest frame of the sphere?
(b) Find the surface charge σ induced on the sphere.
(c) What is the induced dipole moment of the sphere?

45. Faster-Than-Light Motion [500 level]
In 2011, the OPERA experiment reported that $E_\nu = 17$ GeV neutrinos moving from CERN to Gran Sasso appeared to have velocities greater than the speed of light. The measured speed was $v_\nu = (1+\epsilon)c$, with $\epsilon \approx 2.5 \times 10^{-5}$.
(a) Faster-than-light motion could occur for particles with imaginary masses. What negative value of m_ν^2 would be needed to achieve the observed speed?
(b) How fast would an observer need to be moving relative to the Earth to see the neutrinos moving backwards in time in the observer's reference frame?

46. Photon Decay [500 level]
The decay of a photon $\gamma \to e^+ + e^-$ is allowed by charge conservation and angular-momentum conservation. However, it is forbidden by energy-momentum conservation.
(a) For a photon with energy E moving in the z-direction, write down the total four-momentum.
(b) Show that the decay process cannot occur with a massive electron and positron in the final state.
(c) If the daughter particles were massless, would the decay ever be allowed?

47. Dipole Radiation [700 level]
A physical dipole consists of rigid rod of length ℓ with charges q and $-q$ fixed to the ends. The rod lies in the xy-plane and rotates around its center in this plane, with angular frequency ω. As it rotates, it emits radiation.
(a) What condition must ω satisfy if the dipole approximation for the radiation is to be valid?.
(b) What is the power emitted when this condition is satisfied? (You can get partial credit for determining the dependence of the power on the various physical parameters.)

48. Radiating Charge [700 level]
A charge q moves in the xy-plane along a trajectory $x(t) = a\cos\omega t$, $y(t) =$

$a \sin \omega t$. We are interested in the characteristics of the emitted radiation far away, on a sphere of radius R, where $R \gg a$. You will not need the full formulas for the far fields; only the dipole approximation is needed.
(a) How does the intensity of the emitted radiation scale with R, a, and ω?
(b) How does the polarization depend on the polar angle θ? Find the θ-dependence of the electric field amplitudes of E_θ and E_ϕ and their relative phases.
(c) How does the radiation intensity depend on the propagation direction, given by (θ, ϕ)?

49. Radar [700 level]
The discrete nature of rainfall can play an important role in its measurement. The usual method of detection and volume estimation is to measure radar reflected off the rain.
(a) Assume that a raindrop is a sphere of radius a, small compared to the wavelength of radar, and has a frequency-independent dielectric constant ϵ. What is the dominant induced multipole moment in terms of the field $\mathbf{E}$ that the drop finds itself in?
(b) Thus deduce the radar echo intensity as a function of the following:
(i) the raindrop size a
(ii) the radar frequency ω
(iii) the distance R of the rainfall region from the radar station (which emits the radar pulse and then receives the echo).

50. Relativistic Radiation [700 level]
The Larmor formula for the power radiated by an accelerating charge at non-relativistic (NR) velocities is

$$P = \frac{2}{3}\frac{q^2}{c^3}a^2,$$

where the magnitude of the charge is q and the acceleration is a.
(a) Write down a Lorentz-invariant expression which may be the proper relativistic generalization of the Larmor formula. It should reduce to the above in the NR limit and cannot contain anything higher than first derivatives of the velocity. (Hint: The simplest generalization works.)
(b) Show that your expression can be written as

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[\dot{\boldsymbol{\beta}}^2 - \left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)^2\right],$$

whre the dots represent differentiation with respect to the coordinate time t, not with respect to proper time.

(c) Consider 100 GeV electrons in a linear accelerator or in a circular accelerator. Recognizing that the maximum force that can be provided is roughly 50 MeV/m. Use the above expression to find the radiative power loss per meter in both accelerator geometries. Which kind of accelerator loses energy more rapidly?

51. Loops of Charge [500 level]
(a) Find the electric field a distance z above the center of a circular loop of charge with radius R and linear charge density λ.
(b) Find the electric field a distance z above the center of a rectangular loop of charge with sides a and b and linear charge density λ.

52. Overlapping Charged Spheres [500 level]
Two spheres, each of radius R and carrying uniform charge densities ρ and $-\rho$, are placed so that their partially overlap. Call the vector from the positive center to the negative center $\mathbf{d}$, with $d < 2R$.
(a) Find the electric field in the overlap region.
(b) Find the electric field at a distance $r \gg R$.

53. Charged Shell [500 level]
A thick spherical shell $a \leq r \leq b$ carries a charge density $\rho = k/r^2$.
(a) Find the electric field in the three regions: $r < a$, $a < r < b$, and $r > b$.
(b) Find the electrostatic potential in the same three regions.

54. Charged Cone [500 level]
A conical surface (an empty ice cream cone) carries a uniform surface charge σ. The height of the cone is h, as is the radius of the top.
(a) Find the potential difference between the vertex and the center of the top.
(b) Find the electric field along the axis of the cone, on both the convex and concave sides.

55. One-Dimension Lattice [500 level]
Consider an infinite chain of point charges $\pm q$ (with alternating signs), strung out along the x-axis, each a distance a away from its nearest neighbors. Let w be the work *per unit charge* required to position the charges in this fashion
(a) Using dimensional analysis, determine the dependence of w on q, a, and fundamental constants, up to an unknown numerical factor.
(b) Determine the precise expression for w.

56. Electrostatic Pressure [500 level]
A metal sphere of radius R carries a total charge Q.
(a) Find the electrostatic pressure on the charged surface.
(b) What is the force of repulsion between the “northern” and “southern”

hemispheres?
(c) Qualitatively, what effects are responsible for keeping the charge in place on the metal surface?

57. Image Charges [700 level]
A charge q is located a distance r away from the center of a neutral conducting sphere of radius $a < r$.
(a) Find the potential outside the sphere.
(b) Find the surface charge on the sphere.
(c) Find the force between q and the sphere.

58. Charged Cylinder [500 level]
A charge density $\sigma = k \sin 5\phi$ (where k is a constant) is glued over the surface of an infinite cylinder of radius R.
(a) Set up the solution for the potential using separation of variables, and give the equations obeyed by the radial and angular functions.
(b) Find the potential inside the cylinder.
(c) Find the potential outside the cylinder.

59. Charged Disk [500 level]
(a) Find the potential at a point on the axis of a charged disk of radius R, carrying a uniform surface charge σ.
(b) Find the first three terms in the multipole expansion of this potential off the axis, for $r \gg R$.

60. Multipoles [500 level]
A charged ring in the xy-plane (radius R, centered at the origin) carries a uniform line charge λ. Find the first four terms ($\ell = 0, 1, 2, 3$) in the multipole expansion of the potential.

61. Multipoles [700 level]
A charged ring in the xy-plane (radius R, centered at the origin) carries an angle-dependent line charge $\lambda = k \sin \phi$. Find the first three terms ($\ell = 0, 1, 2$) in the multipole expansion of the potential.

62. Separation of Variables [500 level]
A rectangular pipe has its four sides at different potentials. Two opposite sides at $x = \pm b$ are at potential V_0. A third side, at $y = a$ is at V_1, and the fourth side at $y = -a$ is grounded. Find the potential inside the pipe.

63. Equivalent Sources [700 level]
(a) Find the charge density $\sigma(\theta)$ on the exterior surface of a sphere (radius R, centered at the origin) that produces the same electric field, for points exterior

to the sphere, as would a charge q at the point $a < R$ on the positive z-axis.
(b) Show that your answer agrees with expectations as $a \to R$.

64. Orbit in a Dipole Field [500 level]
A stationary electric dipole $\mathbf{p} = p\hat{\mathbf{z}}$ is situated at the origin. A positively charge point particle q (with mass m) executes circular motion (radius r) at a constant speed in the field of the dipole.
(a) Characterize the plane of the orbit.
(b) Find the speed of the particle and its angular momentum.
(c) Find the energy of the particle.

65. Dielectrics [700 level]
A point dipole $\mathbf{p}$ is embedded at the center of a sphere of linear dielectric material (with radius R and dielectric constant κ).
(a) Find the electric potential inside the sphere.
(b) Find the electric potential outside the sphere.

66. Current Flow [500 level]
Consider a current density $\mathbf{J}$ flowing within a region of space Ω and vanishing on the boundary of Ω.
(a) Show that
$$\int d^3r\, \mathbf{J}(\mathbf{r}) = \frac{d\mathbf{p}}{dt},$$
where $\mathbf{p}$ is the total dipole moment.
(b) A rod of length a carries charges q and $-q$ at its ends. The rod rotates end over end around its center with angular frequency ω. Find the resulting current $\mathbf{J}$.

67. Moving Charge [500 level]
Analyze the motion of a particle (charge q, mass m) in the magnetic field of a long straight wire carrying a steady current I.
(a) Is kinetic energy conserved?
(b) Find the force on the particle, in cylindrical coordinates.
(c) Suppose $\dot{z}$ is constant. Write down the equations of motion and show what kind of trajectories are allowed.

68. Relativistic Current Model [500 level]
You may have wondered why, since parallel currents attract, the current within a single wire does not contract into a tiny concentrated stream along the axis. In practice, the current typically distributes itself quite uniformly over the wire. This problem will account for this apparent discrepancy.
(a) Consider a model of the wire in which the positive charges (density ρ_+ are fixed in place, and the negative charges (ρ_-) move at speed v. All these

quantities are independent of the distance from the wire's axis. Show that $\rho_- = -\rho^+\gamma^2$, where γ is the Lorentz factor.
(b) Explain why the moving negative charges do not coalesce.
(c) If the overall wire is charge neutral, where is the compensating charge located?

69. Convective Currents [500 level]
Consider a sphere of radius R carrying a uniform surface charge density σ, which is rotating with angular velocity ω.
(a) Find the magnetic field $\mathbf{B}$.
(b) Is there any electromagnetic energy transport in this system?
(c) Find the force (direction and magnitude) exerted by the "northern" hemisphere on the "southern" hemisphere.

70. Magnetic Monopole [700 level]
Consider the motion of a particle with mass m and electric charge q, moving in the field of a (hypothetical) stationary magnetic monopole g at the origin,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{\mathbf{r}}.$$

(a) Find the acceleration of q, expressing your answer in terms of q, g, m, and the position $\mathbf{r}$ and velocity $\mathbf{v}$ of the charge.
(b) Show that the speed is a constant of the motion.
(c) Show that the vector

$$\mathbf{Q} = m\mathbf{r}\times\mathbf{v} - \frac{\mu_0 q g}{4\pi}\hat{\mathbf{r}}$$

is a constant of the motion.
(d) Choose spherical coordinates, with the polar axis along $\mathbf{Q}$. Calculate $\mathbf{Q}\cdot\hat{\phi}$, and show that θ is a constant of the motion.
(e) Calculate $\mathbf{Q}\cdot\hat{\mathbf{r}}$, and find the magnitude of $\mathbf{Q}$.
(f) Calculate $\mathbf{Q}\cdot\hat{\theta}$, and find $d\phi/dt$ as a function of r.
(g) Express v in spherical coordinates, and use this result to obtain a differential equation for $dr/d\phi$. Solve this equation for the trajectory $r(\phi)$.

71. Helmholtz Coils [500 level]
The magnetic field of a single circular loop of current is far from uniform. You can produce a more nearly uniform field by using two such loops. Consider a configuration with two coaxial loops of equal radii R, carrying equal currents I, and located a distance d apart.
(a) Find the magnitude of the magnetic field on the axis, as a function of the axial position z. Show that $\partial B/\partial z = 0$ at the midpoint between the loops.

(b) Determine d such that the second derivative $\partial^2 B/\partial z^2$ also vanishes at the midpoint.
(c) Find the magnitude of the field at the midpoint for the indicated separation.

72. Convective Currents [500 level]
A thin glass rod of radius R and length L ($R \ll L$) carries a uniform surface charge σ on its lateral surfaces. It is set spinning about its axis with angular velocity ω.
(a) Find the magnetic dipole moment of the rod and the magnetic field at a radial positions $r \gg L$.
(b) Find the magnetic field in the mid-plane of the rod, at a distance $\rho \gg R$ (but allowing for the possibility that $\rho < L$).

73. Broken Magnet [500 level]
An iron rod of length L and square cross section (side $a \ll L$) is given a uniform lengthwise magnetization $\mathbf{M}$ and then bent around nearly into a circle. A narrow gap of width $w \ll a$ remains between the square ends.
(a) Find the bound current distribution in the bend rod.
(b) Find the magnetic field at the center point of the gap between the ends.

74. Motional emf [500 level]
(a) A square loop of wire, with sides of length a, lies in the first quadrant of the xy-plane, with one corner at the origin. In this region, there is a nonuniform time-dependent magnetic field $\mathbf{B}(y,t) = ky^3t^2\hat{\mathbf{z}}$ (where k is a constant). Find the emf induced in the loop.
(b) Suppose the loop is moving with velocity $\mathbf{v} = v_0\hat{\mathbf{y}}$. What is the emf?

75. Capacitance and Resistance [500 level]
(a) Two metal objects are embedded in a weakly conducting material of conductivity σ. Show that the resistance between them is related to the capacitance of the arrangement by $R = \epsilon_0/\sigma C$.
(b) Suppose you connected a battery between the two conductors and charged them up to a potential difference V_0. If you then disconnect the battery, the gradually leaks off; find the voltage between the conductors as a function of time.

76. Cylindrical Current [500 level]
Two long coaxial metal cylinders (radii a and b) are separated by material with a inhomogeneous conductivity $\sigma = k/\rho^2$, for some constant k. Find the resistance between the cylinders.

77. Magnetic Induction [500 level]
An alternating current current $I(t) = I_0 \cos \omega t$ flows down a long straight wire,

and returns along a coaxial conducting tube of radius a.
(a) In what direction does the induced electric field point?
(b) Find $\mathbf{E}(\rho, t)$.
(c) Find the displacement current density $\mathbf{J}_d$.
(d) Integrate the find the total displacement current, and compare it to the conduction current I.

78. Magnetic Induction [700 level]
Imagine an infinitely long cylindrical sheet of uniform resistivity and radius a. A narrow slot runs down one side of the cylinder, breaking it so that it does not have a true circular cross section. The two sides of the slot (at angles $\phi = \pm\pi$) are hooked up to a battery and are maintained at potentials $\pm V_0$, and a steady current flows over the surface, circling around the axis of the cylinder.
(a) Find the potential $\Phi(a, \phi)$ around the sheet, using Ohm's Law.
(b) Determine the potential inside and outside the cylinder.
(c) Find the surface charge density on the cylinder.

79. Superconductor [500 level]
(a) In a perfect conductor, the conductivity is infinite, so $\mathbf{E} = 0$, and any net charge resides on the surface. Show that the magnetic field is time independent inside a perfect conductor.
(b) Show that the magnetic field through a perfectly conducting loop is constant.
(c) A superconductor is a perfect conductor with the additional property that the constant $\mathbf{B}$ inside is in fact zero. Show that the current in a superconductor is confined to the surface.
(d) Superconductivity is lost above a certain critical temperature T_c. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, while cooling it below T_c. Find the induced current density $\mathbf{K}$, as a function of the polar angle θ.

80. Levitating Dipole [700 level]
A familiar demonstration of superconductivity is the levitation of a magnet over a piece of superconducting material. Treat the magnet as a perfect dipole $\mathbf{m}$, a height z above the origin (and constrained to points in the z-direction); pretend the superconductor fills the entire half space below the xy-plane. Because of the Meissner effect, $\mathbf{B} = 0$ in the superconductor.
(a) Find the z-component of the magnetic field just above the superconducting surface.
(b) The boundary condition at $z = 0$ can be satisfied by an image dipole at $-z$. Which way should this image dipole point?

(c) Find the force on the real magnet due to the induced supercurrents in the xy-plane.
(d) The induced current on the surface can be determined from the boundary condition on the tangential component of $\mathbf{B}$. Using the field you get from the image configuration, find the surface supercurrent $\mathbf{K}$.

81. Induction [500 level]
An infinite wire carrying a constant current I in the $\hat{\mathbf{z}}$-direction is moving in the y-direction at a constant speed v.
(a) Find the magnetic field as a function of time.
(b) Find the displacement current at the time $t = 0$, which is the instant when the wire coincides with the z-axis.
(c) Find the induced electric field at $t = 0$.

82. Induced Current [500 level]
A circular wire loop (of radius r and resistance R) encloses a region of uniform magnetic field B, perpendicular to its plane. The field increases linearly with time, $B = \alpha t$.
(a) What is the current in the loop?
(b) An ideal voltmeter is connected between two points $90°$ apart along the loop. What does it read?

83. Electromagnetic Momentum [700 level]
Imagine two parallel infinite sheets, carrying uniform surface charge σ (on the upper sheet at $z = d$) and $-\sigma$ (at $z = 0$). They are moving in the y-direction at a constant speed v.
(a) What is the electromagnetic momentum contained in a region of cross-sectional area A?
(b) Now suppose the top sheet moves slowly down (with speed u) until it reaches the bottom sheet, so the fields disappear. By calculating the magnetic force on the charge $q = \sigma A$, find the impulse delivered to the sheet during the entire process. Compare the result to the answer from part a.

84. Charged Ferromagnet [500 level]
Imagine an iron sphere of radius R that carries a charge Q and uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$.
(a) Compute the angular momentum stored in the electromagnetic fields.
(b) Suppose the sphere is gradually (and uniformly) demagnetized (perhaps by heating it past the Curie point). Determine the electric field during the demagnetization.
(c) Find the torque that the induced $\mathbf{E}$ exerts on the sphere and the total angular momentum imparted to sphere during the course of the demagnetization.

(d) Suppose that instead of demagnetizing the sphere, we discharge it, by connecting a grounded wire to the north pole. Assume the current flows over the surface in such a way as to keep the distribution uniform. Determine the torque on the sphere and the total angular momentum imparted during the discharge.

85. Field Energy [700 level]
An infinitely long cylindrical tube of radius a moves at a constant speed v along its axis. It carries a net charge per unit length λ, uniformly distributed over its surface. Surrounding it, at radius b, is another cylinder, moving with the same velocity but carrying an opposite charge.
(a) Find the energy per unit length stored in the fields.
(b) Find the momentum per unit length in the fields.
(c) Find the energy per unit time transported by the fields across a plane perpendicular to the cylinders.

86. Double Dipole [500 level]
A sphere of radius R carries a uniform polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ (not necessarily in the same direction).
(a) Find the electromagnetic momentum density of this configuration.
(b) Find the total momentum stored in the fields.

87. Wave Absorption [500 level]
Consider a particle of charge q and mass m, free to move in the xy-plane in response to an electromagnetic plane wave propagating in the z-direction.
(a) Initially ignoring the magnetic force, find the velocity of the particle as a function of time. (Assume the initial conditions are such that the average velocity is zero.)
(b) Now calculate the magnetic force on the particle.
(c) Calculate the time average of the magnetic force.
(d) For energy to be absorbed, there needs to be some resistance to the motion of the charge. Include a force of the form $-\gamma m\mathbf{v}$, for some damping constant γ. Repeat the calculations from part a (ignoring the exponentially damped transient) and part b.

88. Waveguide [700 level]
Consider a rectangular waveguide with conducting walls and dimensions a and b. Consider the TE_{mn} mode.
(a) Calculate the group velocity for this mode.
(b) Find the time averaged Poynting vector $\langle\mathbf{S}\rangle$ and energy density $\langle u\rangle$.
(c) Determine the energy per unit time and per unit length carried by the wave, and show that the energy moves at the group velocity.

89. Relativistic Fields [700 level]
A parallel-place capacitor, at rest in the frame S_0 and tilted at a 45° degree angle to the x-axis, carries charge densities $\pm\sigma_0$ on the two planes. The frame S is moving in the x-direction at a speed v relative to S_0.
(a) Find $\mathbf{E}_0$, the field in S_0.
(b) Find $\mathbf{E}$, the field in S.
(c) What angle do the plates make with the x-axis in the S frame?
(d) Is the field perpendicular to the plates in the S frame?

90. Relativistic Fields [700 level]
A charge q is released from rest at the origin, in the presence of a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$ and uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$ (with $E_0 < cB_0$).
(a) Boost to a frame in which the elecric field vanishes, and find the field in this frame.
(b) Find the trajectory of the charge in the boosted frame.
(c) Return to the original frame, and determine the trajectory in that frame.

91. Relativistic Fields [700 level]
Using the field strength tensor $F^{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu}$, demonstrate that:
(a) If $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in one Lorentz frame, then they are perpendicular in all Lorentz frames.
(b) If $|\mathbf{E}| > c|\mathbf{B}|$ in one Lorentz frame, then $|\mathbf{E}| > c|\mathbf{B}|$ in all frames.

92. Antenna [700 level]
A thin, straight, conducting wire is centered on the origin, oriented along the z-axis, and carries current $\mathbf{I} = I_0\hat{\mathbf{z}}\cos\omega_0 t$ everywhere along its length l. Define $\lambda_0 \equiv 2\pi c/\omega_0$.
(a) What is the electric dipole moment of the wire?
(b) What are the scalar and vector potentials everywhere outside the source region (at a distance $r \gg l$)? State your choice of gauge, and make no assumptions about λ_0.
(c) Consider the potentials in the regime $r \gg l \gg \lambda_0$. Qualitatively describe the radiation pattern and compare it to the standard dipole case, where $r \gg \lambda_0 \gg l$.

93. Magnetic Material [700 level]
A right circular cylinder of radius R, length L, and uniform mass density ρ has a uniform polarization $\mathbf{M}$ parallel to its axis. If it is placed below an infinitely-permeable ($\mu \to \infty$) flat surface, it is found to stick for some lengths $L \gg R$. What is the maximum length L such that the magnetic force prevents the cylinder from falling due to gravity?

94. Relativistic Acceleration [700 level]
A rocket ship accelerates away from the Earth at a constant acceleration g (in the rocket's frame).
(a) Calculate the angular size of the Earth as viewed from the rocket, as a function of the proper time on the rocket.
(b) The angular size in part (a) approaches a limit as proper time gets large. What is it?

95. Resistive Networks [500 level]
The edges of a cube consist of equal resistors of resistance R, which are joined at the corners.
(a) Let a battery be connected to two opposite corners of the cube. What is the effective resistance?
(b) Let a battery be connected to two opposite corners of *one side of the cube.* What is the effective resistance?

96. Variable Capacitor [500 level]
A variable capacitor is connected to a battery of emf $\mathcal{E}$. The capacitor initially has a capacitance C_0 and charge q_0. The capacitance is caused to change with time so that the current I is constant.
(a) Calculate the power supplied by the battery.
(b) Compare the result of part (a) with the time rate of change of the energy stored in the capacitor, and account for any difference.

97. Capacitor [500 level]
Calculate the capacitance C of a spherical capacitor of inner radius a and outer radius b, which is filled with an inhomogeneous dielectric varying as $\epsilon = \epsilon_0 + \epsilon_1 \cos^2 \theta$.

98. Magnetic Mirror [700 level]
An ion moves in a helical path around the axis of a long solenoid wound so that the ion encounters a region in which the field intensity increases from B_1 to B_2. Under what circumstances will the ion be reflected?

99. Total transmission [500 level]
A plane electromagnetic wave is normally incident from vacuum on a plane surface film of uniform thickness d, covering a semi-infinite dielectric. Assume $\mu = \mu_0$ for both media, and that the film and substrate have indices of refraction n_1 and n_2, respectively.
(a) Find an expression for the wave reflected back into the vacuum in terms of d, n_1, n_2, and the vacuum wavelength λ_0.
(b) Under what conditions will the reflected wave vanish?

100. Diffraction Grating [500 level]
A diffraction grating is made up of N narrow slits, each half as wide as the previous one. The slits are separated by a distance d. What is the angular distribution of the intensity of light of wavelength λ passing through the grating?

101. Polarizers [500 level]
Light is passed through an array of perfect polarizers. The polarization planes are approximately lined up, but there is a random error between each successive pair of polarizers, with a Gaussian distribution $N \exp(-a\theta^2)$ for the relative angular deviation θ. Find the average attenuation coefficient per polarizer (after the first) for a beam of light passing through. Assume $a \gg 1$.