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Proper-Time Quantum Mechanics. II

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This paper is a continuation of a previous investigation of a proper-time formulation of quantum mechanics based upon an extension of the Poincaré algebra to include a four-vector position operator. The discrete operations of space and time inversion and particle conjugation are studied as groups of automorphisms which must also be supported by the representations of the extended algebra. Supporting representations and matrix forms of the inversions are studied in the content of a proper-time framework. Representations of the extended algebra are decomposed with respect to their constituent irreducible Poincaré representations, and for unique spin we connect our representations (fields) with those of Weinberg. The basis which exhibits the Poincaré content of a representation is studied, and the transformation to that basis is connected for spin $\frac{1}{2}$ to the Foldy-Wouthuysen transformation. The unique spin-$\frac{1}{2}$ representation of the extended algebra has a decomposition which accomodates the leptons in a very suggestive way. The leptons are distinguished from one another without "internal" quantum numbers. The physical mass spectrum (0, m_e, and m_μ) is both allowed and unexplained. The electromagnetic and weak currents assume a simple form. Certain aspects of the formulation of a proper-time field theory are also discussed.

I. INTRODUCTION

IN a previous investigation,[1] we extended the Poincaré algebra to a larger algebra[2] including a relativistic position operator X^μ. Thus the time X^0 is treated on an equal footing with the spatial operator for position, $\mathbf{X}$. The extended algebra (referred to as the *XPM* algebra) and the X^μ operator were defined by the following commutation rules:

$$[P^\mu,X^\nu]=ig^{\mu\nu}\,,$$
$$[X^\mu,X^\nu]=0\,,$$
$$[P^\mu,P^\nu]=0\,,$$
$$[M^{\mu\nu},X^\lambda]=i(g^{\lambda\nu}X^\mu-g^{\lambda\mu}X^\nu)\,,$$
$$[M^{\mu\nu},P^\lambda]=i(g^{\lambda\nu}P^\mu-g^{\lambda\mu}P^\nu)\,,$$
$$[M^{\mu\nu},M^{\rho\sigma}]=-i(g^{\mu\rho}M^{\nu\sigma}+g^{\nu\sigma}M^{\mu\rho}-g^{\nu\rho}M^{\mu\sigma}-g^{\mu\sigma}M^{\nu\rho})\,,$$

where P^μ and $M^{\mu\nu}$ generate translations and Lorentz transformations. When realized as a set of operators, the X^μ, P^μ, and $M^{\mu\nu}$ are to be interpreted as the physical four-position, four-momentum, and four-tensor of total angular momentum at a given instant of proper time. The representation space of the algebra is to be interpreted as the kinematical specification of the state of the system at a given instant of proper time. The proper-time dynamics $U(\tau_2,\tau_1)=e^{iH(\tau_2-\tau_1)}$ is to give a continuous automorphism of the algebra (Heisenberg picture) or the representation space (Schrödinger picture). The use of an invariant Hamiltonian H as the generator for proper-time translations maintains a manifest covariance for the theory and keeps the time (X^0) and space operators on an equal footing. As with the classical relativistic mechanics which we mimic, the proper time is a formal device for covariance and is eventually eliminated from the calculation. The kinematical specification of states which we found was more general than the standard theory since it included states which were superpositions of various masses which in configuration space appear as wave packets spread out in space-time where $\int d^4y\,\psi^*\psi=1$. The standard theory is retrieved in the limit of mass eigenstates for free particles.

In this paper we continue this investigation in several important aspects. We first study space and time inversion and charge conjugation. The inversions form a discrete group of four elements (including the identity) and are defined below as a group via a product table with the restriction that they have definite commutation rules with the *XPM* algebra. One is then able to view the inversions as a discrete group of automorphisms of the *XPM* algebra (i.e., a mapping of the algebra into itself which preserves the structure constants). Thus in order to have fields upon which the inversions are well defined, it follows that the representation space of the *XPM* algebra must also serve as a representation space of the inversion group. Consequently, Sec. II is devoted to finding the (generally reducible) *XPM* representations supporting inversions and to finding the matrix form of the inversions on the resulting space. The inversions are found to be represented by unitary operators (because the structure constants are unaltered) which either mutually commute or anticommute. Our approach rests heavily on a similar treatment by Gel'fand *et al.*[3] with respect to the homogeneous Lorentz group. The bilinear forms which are invariant with respect to inversions and the transformations generated by the *XPM* algebra are found. By inserting various operators in the invariant forms, one is able

[1] Joseph E. Johnson, Phys. Rev. **181**, 1755 (1969), hereafter referred to as I; thesis, State University of New York at Stony Brook, 1967 (unpublished; paper presented at the November 1967 New York APS meeting (unpublished). Our notation is the same as that in I, with $\hbar=c=1$, $x^0=t$, $P^0=E$, and $g^{00}=1=-g^{ii}$. We use $\epsilon(\alpha)=+1$ ($\alpha>0$), 0 ($\alpha=0$), and -1 ($\alpha<0$).

[2] Both H. E. Moses, Ann. Phys. (N.Y.) **52**, 444 (1969) and J. J. Aghassi, P. Roman, and R. M. Santilli, Phys. Rev. D **1**, 2753 (1970) contain treatments closely related to our approach in I. The approach of Aghassi *et al.* has been criticised by M. Noga, Phys. Rev. D **2**, 304 (1970). See note added in proof.

[3] I. M. Gel'fand, R. A. Minlos, and Z. Yu. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (MacMillian, New York, 1963).

TABLE I. Inversion group products. The representations of these operators have the same product table if they commute. But if they are to anticommute, the quantities prefaced by * are to be taken with a minus sign.

	I	I_s	I_t	I_{st}
I	I	I_s	I_t	I_{st}
I_s	I_s	I	I_{st}	I_t
I_t	I_t	$*I_{st}$	I	$*I_s$
I_{st}	I_{st}	$*I_t$	I_s	$*I$

to construct other forms, such as currents and Poincaré-invariant bilinear forms.

The concept of particle conjugation is studied as another automorphism of the XPM algebra which leaves the structure constants unaltered but which performs complex conjugation and thus is represented by an antiunitary operator. One may consider particle conjugation and the identity as a group of two elements.

Since the Poincaré algebra is contained as a subalgebra of the XPM algebra, it follows that the XPM representations (which support inversions) are also Poincaré representations which are in general reducible. As the fundamental particles are identified with Poincaré representations, one needs a decomposition of the representation with respect to the Poincaré algebra. This is then accomplished by seeking a new basis for the representation space labeled partly as an irreducible Poincaré representation is labeled. The remainder of the labeling consists of Poincaré covariants. The resulting investigation ties together a number of ideas of the standard theory in a simple fashion: The various wave equations and subsidiary conditions are seen to be eigenvalue equations for these Poincaré-covariant operators. The imposition of wave equations becomes a requirement that a physical particle belongs within a Poincaré-invariant subspace of the XPM representation space. The transformation between the old basis and the Poincaré basis constitutes a solution of the wave equation. For spin $\frac{1}{2}$ this transformation contains the Foldy-Wouthuysen transformation as one portion of the basis transformation as seen on part of the representation space. The operator X^μ, when viewed on this portion of the space, becomes a direct four-vector generalization of the Foldy-Wouthuysen three-vector operator (since their spatial parts are identical). We discuss this basis for the unique spin representations and connect our framework with Weinberg's theory for the corresponding spin. We present a detailed discussion of this program for spin $\frac{1}{2}$, where the Poincaré basis is defined by the operators $\epsilon(\gamma\cdot P)$, $\epsilon(P^0)$, m, $\mathbf{k}$, and helicity w^0. It is found that for spin $\frac{1}{2}$ the XPM representation space consists of a fourfold infinity of irreducible unitary Poincaré representations determined by the four sign combinations of P^0 and $\gamma\cdot P$ for each mass m from zero to infinity.

In Sec. III we point out that the leptons can be placed in this one XPM spin-$\frac{1}{2}$ representation. Positive and negative energy are associated with particle and antiparticle of the same mass while $\gamma\cdot P=m\epsilon(\gamma\cdot P)$ has the spectrum m_e, 0, $-m_\mu$ for the electron, neutrinos, and muon. This spectrum for $\gamma\cdot P$ is allowable within this framework but remains unexplained. The electromagnetic and weak currents assume an appealing form and the leptons are distinguished from one another without additional internal quantum numbers or degrees of freedom.

II. DISCRETE AUTOMORPHISMS AND TOTALLY INVARIANT BILINEAR FORMS

There are certain discrete mappings (or automorphisms) which carry the elements of the XPM algebra of observables into a new set with the same structure constants. Those automorphisms of particular interest are the space and time inversions and charge conjugation since they can be more or less understood physically in terms of inverted spatial axis, inverted sequences of events, and the interchange of antiparticle and particle. In the usual treatment these automorphisms are defined in such a way as to give the intuitive results of such operations. Such an approach is also dictated by the mathematical framework of the standard theory. For example, time inversion changes X^0 but leaves the energy P^0 unchanged since P^0 is taken as the Hamiltonian and must retain the physically admissible spectrum $P^0\geq 0$. Furthermore, reversal of the time sequence of events means that $\mathbf{P}\rightarrow -\mathbf{P}$ while $\mathbf{X}\rightarrow \mathbf{X}$, with the result that time inversion alters the sign of the commutation rule $[X^i,P^i]=i\delta^{ij}$. Thus time inversion must be represented by an antiunitary operator. These arguments are generally familiar.

Within a framework using the XPM algebra and a proper-time dynamics, a slightly different approach suggests itself. The automorphisms can be most symmetrically approached by considering a single antiunitary operation closely related to complex conjugation. It will be seen to be equivalent to particle conjugation and performs a certain (antiunitary) automorphism defined below on the algebra. Separately from this automorphism we can define an inversion group of four elements: the identity I, space inversion I_s, time inversion I_t, and space-time inversion I_{st}. This discrete group of four automorphisms leaves the XPM structure constants unchanged, and they can be represented as unitary operators on supporting XPM representations. This separation of the problem is advantageous from both a mathematical and a physical point of view when using a proper-time dynamics. Other automorphisms may be formed which are products of these automorphisms. We will first study the inversion group and find the matrix form of the operators and the supporting representations.

The group multiplication table for space (I_s) time (I_t), and space-time (I_{st}) inversion operators is given in Table I (with the identity represented by I). Within a proper-time formalism the most natural definitions for the commutators of the inversions with the XPM

generators are given in Table II. Those rules involving $M^{\mu\nu}$ and X_μ are the same as one traditionally uses. The angular momentum tensor $L^{\mu\nu}=X^\mu P^\nu - X^\nu P^\mu$ and spin tensor $S^{\mu\nu}=M^{\mu\nu}-L^{\mu\nu}$ are transformed under inversions in the same way as $M^{\mu\nu}$. The distinguishing feature of our definitions is that we take the inversion of P^μ to be the same as X^μ thus maintaining their basic symmetry. This is possible within a proper-time dynamics and is consistent with our requirement that the proper time τ and mass $m=+\sqrt{(P_\mu P^\mu)}$ are to be unchanged by these inversions. We see that the physical subspace $m\geq 0$, $m^2\geq 0$ is mapped into itself under inversions, and the free-particle motion in the Heisenberg picture, $dX^\mu/d\tau=P^\mu/m$, is inversion invariant. One observes that when τ is eliminated, one gets the traditional inversion properties for the standard three-velocity, dX^i/dX^0, and three-momenta, P^i/P^0. This is because a negative-energy particle in a proper-time framework which has $P^\mu=(-|P^0|, \mathbf{P})$ actually represents motion in the direction $-\mathbf{P}=\epsilon(P^0)\mathbf{P}$ with spin $\epsilon(P^0)S^{\mu\nu}$.

We are interested in finding those representations of the XPM algebra which will support the inversion group, and in finding the matrix form of the inversions on the generally reducible supporting representations. In I we found all XPM representations by taking the direct product of the von Neumann algebra (XP) representation with the homogeneous Lorentz algebra $(S^{\mu\nu})$ representations. Thus the action of any inversion on the XPM representation will be the direct product of the matrix forms of the inversions on the XP and $S^{\mu\nu}$ representations. We thus study the inversions on the separate subalgebras.

We first consider the inversions on the XP algebra. It readily follows from the commutation rules that

$$I_s|a^0,a^i\rangle=\eta_s|a^0,-a^i\rangle\,,$$
$$I_t|a^0,a^i\rangle=\eta_t|-a^0,a^i\rangle\,,$$
$$I_{st}|a^0,a^i\rangle=\eta_{st}|-a^0,-a^i\rangle\,,$$

where a^μ represents the eigenvalues of either P^μ or X^μ. Since the square of an inversion is the identity, then $\eta=\pm 1$ for each η. Furthermore, since $I_sI_t=I_{st}$ then $\eta_{st}=\eta_s\eta_t$. The bilinear invariant form

$$\langle a'^\mu|a^\mu\rangle=\delta^4(a'^\mu-a^\mu)$$

remains invariant under all inversions as well as the completeness relations. It is easily checked that the inversions are both unitary and Hermitian on this scalar product.

We now consider the inversions acting upon a representation space of the homogeneous Lorentz algebra. Gel'fand *et al.*[3] contains a thorough discussion of these representations. We briefly summarize their results not only for completeness, but also because one obtains slightly different results when the XP algebra is adjoined due to the use of a proper-time dynamics. They show that the two-valuedness of some $S^{\mu\nu}$ representations can lead to a two-valuedness of the representation of the inversion group. This has the consequence that the matrices representing the inversions can either mutually commute or anticommute and still form a representation of the group products in Table I (to within a sign). In each case, of course, the inversions must have commutators with $S^{\mu\nu}$ given by Table II. We will use the same symbol for the abstract group element and its matrix representation (although they may differ by a sign).

TABLE II. A positive or negative sign indicates whether the two operators indicated in that row and column are to anticommute or commute, respectively.

	X^0	X^i	P^0	P^i	S^{ij}	S^{0i}	b_0	b_1	γ^0	γ^i	$\gamma\cdot P$	w^0	γ^5
I_s	−	+	−	+	−	+	−	+	−	+	−	+	+
I_t	+	−	+	−	−	+	−	+	+	−	−	−	+
I_{st}	+	+	+	+	−	−	−	−	+	+	−	+	−
K_c	−	−	+	+	+	+	−	+	+	+	−	−	−

We first take the case where the inversions mutually commute. Then either $I_{st}=+I$ with $I_t=I_s$ or $I_{st}=-I$ with $I_t=-I_s$. For each of these possibilities, I_s is determined by whether the representation is self-conjugate or not.

Case I. If the representation is self-conjugate ($b=\dot{b}$ and thus either $b_0=0$ or $b_1=0$) then there are two inequivalent representations (distinguished by η_s) given by $I_s|b_0,b_1,s,\sigma\rangle=\eta_s(-1)^{s-b_0}|b_0b_1,s,\sigma\rangle$, with $\eta_s=+1$ or $\eta_s=-1$.

Case II. If the representation is not self-conjugate ($b\neq\dot{b}$ and thus neither $b_0=0$ nor $b_1=0$) then there is one representation given on the reducible sum space $|b,s,\sigma\rangle\oplus|\dot{b},s,\sigma\rangle$ by

$$I_s|b,s,\sigma\rangle=(-1)^{s-b_0}|\dot{b},s,\sigma\rangle\,,$$
$$I_s|\dot{b},s,\sigma\rangle=(-1)^{s-b_0}|b,s,\sigma\rangle\,.$$

If we define a new basis by

$$|b_+\rangle=|b\rangle \quad\text{and}\quad |b_-\rangle=(-1)^{s-b_0}|\dot{b}\rangle\,,$$

then I_s becomes $I_s|b_\pm,s,\sigma\rangle=|b_\mp,s,\sigma\rangle$. Still another commuting representation is discussed by Gel'fand *et al.* if the space is doubled.

If the matrices representing inversions anticommute then there are just two cases depending upon whether the representation is self-conjugate or not.

Case III. If the representation is self-conjugate, then one forms a sum space by adding another identical representation $|b_a\rangle\oplus|b_b\rangle$, where a and b are used to distinguish the two identical representations. By choosing the basis $|b_+\rangle=|b_a\rangle+i|b_b\rangle$, $|b_-\rangle=(-1)^{s-b_0}(|b_a\rangle-i|b_b\rangle)$, the inversions take the form

$$I_s=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad I_t=\begin{pmatrix}0&-i\\i&-0\end{pmatrix},\quad I_{st}=\begin{pmatrix}+i&0\\0&-i\end{pmatrix}$$

on the space $\begin{pmatrix}|b_+\rangle\\|b_-\rangle\end{pmatrix}$.

Case IV. If the representation is not self-conjugate then one forms the sum space $|b\rangle\oplus|\dot{b}\rangle$. Using the basis defined by $|b_+\rangle=|b\rangle$, $|b_-\rangle=(-1)^{s-b_0}|\dot{b}\rangle$ the inversions take the same form as in case III.

We can now find the form of the inversions when acting on an XPM representation by taking the product of the inversions on the separate XP and $S^{\mu\nu}$ representation spaces. At the same time we will discuss the invariant bilinear form for each case when the spin representation is finite dimensional. For case I we have

$$I_s|a^0,a^i,b,s,\sigma\rangle=\eta_s|a^0,\,-a^i,\,b,\,s,\,\sigma\rangle\,,$$
$$I_t|a^0,a^i,b,s,\sigma\rangle=\eta_t|-a^0,\,a^i,\,b,\,s,\,\sigma\rangle\,,$$
$$I_{st}|a^0,a^i,b,s,\sigma\rangle=\eta_s\eta_t|-a^0,\,-a^i,\,b,\,s,\,\sigma\rangle\,;$$

where a^μ represents the eigenvalues of either P^μ or X^μ and where, with independent signs, $\eta_s=\pm(-1)^{s-b_0}$ and $\eta_t=\pm(-1)^{s-b_0}$ and thus $\eta_s\eta_t=\pm1$. The bilinear form invariant under both XPM and inversion transformations is given by $\langle a'^\mu,b',s',\sigma'|\eta|a^\mu,b,s,\sigma\rangle=(-1)^{s-b_0}\times\delta^4(a'^\mu-a^\mu)\delta_{b'b}\delta_{s's}\delta_{\sigma'\sigma}$. We use the symbol η with no subscript for the metric. Since b is self-conjugate and is to define a finite representation, b_0 must be zero and b_1 must be an integer. Thus the factor $(-1)^{s-b_0}=(-1)^s$ gives an indefinite scalar product except in the $b_1=1$ case of unique spin zero.

For case II we get the commuting inversions:

$$I_s|a^0,a^i,b_\pm,s,\sigma\rangle=\eta_s|a^0,\,-a^i,\,b_\mp,\,s,\,\sigma\rangle\,,$$
$$I_t|a^0,a^i,b_\pm,s,\sigma\rangle=\eta_t|-a^0,\,a^i,\,b_\mp,\,s,\,\sigma\rangle\,,$$
$$I_{st}|a^0,a^i,b_\pm,s,\sigma\rangle=\eta_s\eta_t|-a^0,\,-a^i,\,b_\pm,\,s,\,\sigma\rangle\,,$$

where η_s and η_t are independently ±1. The totally invariant bilinear form is

$$\langle a'^\mu,b_\pm',s',\sigma'|\eta|a^\mu,b_\pm,s,\sigma\rangle=\delta^4(a'^\mu-a^\mu)\delta_{b_\mp',b_\pm}\delta_{s',s}\delta_{\sigma',\sigma}\,.$$

For cases III and IV we have

$$I_s|a^0,a^i,b_\pm,s,\sigma\rangle=\eta_s|a^0,\,-a^i,\,b_\mp,\,s,\,\sigma\rangle\,,$$
$$I_t|a^0,a^i,b_\pm,s,\sigma\rangle=\pm i\eta_t|-a^0,\,a^i,\,b_\mp,\,s,\,\sigma\rangle\,,$$
$$I_{st}|a^0,a^i,b_\pm,s,\sigma\rangle=i\eta_s\eta_t|-a^0,\,-a^i,\,b_\pm,\,s,\,\sigma\rangle\,.$$

The bilinear form which is invariant under XPM transformations and spatial inversions is the same as the bilinear invariant in case II. However, this form is not invariant under I_t and I_{st}, because it changes sign under each. In order to get a totally invariant bilinear form, we need to multiply the metric by an operator which commutes with all operators except I_t and I_{st}. With these it must anticommute. Although no operator satisfies this requirement exactly, the sign of the energy $\epsilon(P^0)$ satisfies it except when operating on states with $P^0=0$. For physical particles this does not appear to entail problems except possibly for $m=0$ states in the limit of zero momenta. Thus we will use the totally invariant bilinear form (on the momentum basis)

$$\langle k'^\mu b_\pm's'\sigma'|\eta|k^\mu,b,s,\sigma\rangle=\epsilon(k^0)\delta^4(k'^\mu-k^\mu)\delta_{b_\mp',b_\pm}\delta_{s's}\delta_{\sigma'\sigma}.$$

In the limit of $P^0=0$ this gives a zero norm.

One can also consider the automorphism K_c of particle-antiparticle conjugation. We define K_c by the following mapping of the algebra $i\to-i$, $X^\mu\to X^\mu$, $P^\mu\to-P^\mu$, $M^{\mu\nu}\to-M^{\mu\nu}$ and, consequently, $L^{\mu\nu}\to-L^{\mu\nu}$, $S^{\mu\nu}\to-S^{\mu\nu}$. The operation is easily pictured when the spin representation is trivial (i.e., no spin) by acting on the position diagonal representation with an operator which just performs complex conjugation. For finite-dimensional $S^{\mu\nu}$ representations the invariant b_1 is real. From the equations $\frac{1}{2}S_{\mu\nu}S^{\mu\nu}=\mathbf{S}^2-\mathbf{R}^2=b_0^2+b_1^2-1$ and $\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma}=\mathbf{S}\cdot\mathbf{R}=-ib_0b_1$, it follows that since $S^{\mu\nu}\to-S^{\mu\nu}$ under K_c, b_1 must change sign. The matrix which accomplishes this can be worked out for individual cases by simply observing the commutation or anticommutation rules of K_c with the operators whose eigenvalues label the basis of interest.

III. POINCARÉ CONTENT—WEINBERG'S FORMALISM

The Poincaré (PM) algebra is a subalgebra of the XPM algebra and thus the XPM representations form (generally reducible) Poincaré representations. The elementary particles (or fields) are associated with the irreducible Poincaré representations. Thus it is of interest to choose a new basis for the XPM representations which exhibits the irreducible Poincaré representations contained in it. This can be done by choosing a new representation space to be an eigenstate of commuting Poincaré invariants. We will see that the selection of this basis (i.e., finding the Poincaré invariants) in each XPM representation is equivalent to a study of all possible relativistic wave equations and restrictions by subsidiary conditions, at least within the standard framework. Furthermore, finding the transformation matrices to the new basis will be equivalent to the solution of the "wave equation." Before discussing the general problem any further we will consider the special but important case of unique spin representations.

The unique spin representations of the XPM algebra which support the inversion group are spanned by the basis vectors $|k^\mu,b,s,\sigma\rangle\oplus|k^\mu,\dot{b},s,\sigma\rangle$, where $\dot{b}$ denotes the conjugate representation. We have used the momentum representation with k^μ (the eigenvalue of P^μ) ranging over all physical four-momenta. Instead of k^μ one may use the set $k=+\surd(k_\mu k^\mu)$, $\epsilon(k^0)$, and $\mathbf{k}$. Because we are dealing with a unique spin representation, $b_0=s$, $b_1=s+1$, and the conjugate representation $\dot{b}$ is given by $b_0=s$, $b_1=-(s+1)$. These two representations are distinguished for a given s by the sign of b_1. Thus for each mass m there are four separate parts to the space given by the four sign combinations of $\epsilon(k^0)$ and $\epsilon(b_1)$. However, b_1 is not a Poincaré scalar, but rather a pseudoscalar, since it changes sign under both space and time inversions. One would guess from the fourfold nature of the space that it contains exactly four irreducible Poincaré representations. This is a true result but to show it we will find a Poincaré scalar which is inversion invariant to replace b_1. The new scalar would be expected to have two and only two eigenvalues because of the multiplicity of $\epsilon(b_1)$.

This problem was solved by Weinberg[4] from quite a different approach by building up fields from irreducible Poincaré representations. Several such representations were needed in order to support the inversion group. After his building process was complete, the field which resulted had twice as many components as desired. The desired number of irreducible Poincaré representations was two, corresponding to particle and antiparticle. Thus one needed a constraint to reduce the number of components from four to two. This constraint equation becomes a wave equation for local fields and in momentum space becomes a constraint upon a Poincaré invariant reducing the number of components by half. This invariant is precisely the one we seek. Weinberg proved that the unique spin representation space supports a tensor operator $\gamma^{\mu_1\mu_2\cdots\mu_{2s}}$ which can be constructed from the bilinear forms on the space. It is a totally symmetric traceless tensor which is a generalization of the Dirac matrices. It trivially follows from his Eq. (B10) and the supporting equations $\gamma^{\mu_1\mu_2\cdots\mu_{2s}}\gamma^{\nu_1\nu_2\cdots\nu_{2s}}P_{\mu_1}P_{\mu_2}\cdots P_{\mu_{2s}}P_{\nu_1}P_{\nu_2}\cdots P_{\nu_{2s}} = m^{4s}$ that $\gamma_{\mu_1\mu_2\cdots\mu_{2s}}P^{\mu_1}P^{\mu_2}\cdots P^{\mu_{2s}} = \pm m^{2s}$. Intuitively one sees the sign multiplicity most clearly by going to the rest frame. For the case of spin $\frac{1}{2}$ this becomes $\gamma_\mu P^\mu = \pm m$. Thus his wave equation (7.19) is a constraint which eliminates half of the representation space.

One is reminded that in Weinberg's formalism, as in the standard approach, that these restrictions contain the dynamics of the free fields. In our approach, the equations involving $\gamma_{\mu_1\mu_2\cdots\mu_{2s}}P^{\mu_1\mu_2\cdots\mu_{2s}}$ are kinematical restrictions which are in principle no different than an equation restricting the helicity to be positive or negative. The dynamics is given by the proper-time development operator $e^{iH\tau}$. It is only in the special case of a mass eigenstate without interactions that one can reduce the dynamics to this form.

Returning now to the point of view of this paper, we see that $\gamma_{\mu_1\mu_2\cdots\mu_{2s}}P^{\mu_1}P^{\mu_2}\cdots P^{\mu_{2s}}$ is the Poincaré invariant we seek. The basis which makes the Poincaré content of the XPM representation explicit is

$$|k,\epsilon(k^0),\epsilon(\gamma_{\mu_1}\cdots\gamma_{\mu_{2s}}P^{\mu_1}\cdots P^{\mu_{2s}}),s,\mathbf{k},w^0\rangle,$$

where $w^0 = \mathbf{S}\cdot\mathbf{P}$ is the helicity, and must be used instead of σ, the third component of spin, because the spin S^3 does not commute with $\gamma_{\mu_1\cdots\mu_{2s}}P^{\mu_1}\cdots P^{\mu_{2s}}$. Thus each unique spin XPM representation contains a fourfold infinity of irreducible Poincaré representations. The wave equations of Weinberg are thus seen as eigenvalue equations showing in which Poincaré subspace a given vector lies. The subspaces $|\pm\rangle$ of course satisfy the equations $(\gamma_{\mu_1\cdots\mu_{2s}}P^{\mu_1}\cdots P^{\mu_{2s}}\mp m^{2s})|\pm\rangle = 0$. One would expect that the two signs of the energy $\epsilon(P^0)$ correspond to representations denoting particle and antiparticle. The two signs of $\gamma_{\mu_1\cdots\mu_{2s}}P^{\mu_1}\cdots P^{\mu_{2s}}$ have no interpretation at this point except to imply that they correspond to two different particles. We will return to this point in Sec. IV. In order to make the nature of these subspaces more lucid we will continue our discussion with the familiar case of $S=\frac{1}{2}$.

For spin $\frac{1}{2}$, the Poincaré content is exhibited with the basis $|k,\epsilon(k^0),\epsilon(\gamma_\mu P^\mu),\mathbf{k},w^0\rangle$. Half of the space $\epsilon(\gamma\cdot P) = +1$ satisfies the Dirac equation $(\gamma\cdot P - m)|\gamma\cdot P>0\rangle = 0$, while the other half satisfies $(\gamma\cdot P+m)|\gamma\cdot P<0\rangle = 0$. Each subspace is mapped into itself by both the Poincaré group and charge conjugation, and by the inversions. The metric, discussed previously, becomes $\eta = \gamma^0\epsilon(P^0)$ for an anticommuting inversion group. The space is actually an eight-component space for each value of the mass $k\geq 0$ and three-momenta $\mathbf{k}$. The operator X^μ does not commute with $\epsilon(\gamma\cdot P)$ or with w^0. Thus the most natural basis for a localized state is either $|y^\mu,\epsilon(b_1),\sigma\rangle$ or $|y^\mu,\gamma^0,\sigma\rangle$. The first of these is the familiar γ_5-diagonal representation since $\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma} = ib_0b_1 = \frac{3}{4}\gamma_5$ and thus $\epsilon(b_1) = -i\gamma_5$. The other is the familiar γ^0-diagonal representation which is useful because of the form of the metric. The various operators are written out in these representations in the Appendix. If we have a given particle, say, an electron, then it is to be associated with an irreducible Poincaré representation with a given mass m_e. By convention it is taken to be the positive energy $\gamma\cdot P>0$ subspace while the positron is the negative energy $\gamma\cdot P>0$ subspace (the two being related by charge conjugation). In the momentum representation it is then obvious that the particle corresponds to one irreducible Poincaré representation and likewise for its antiparticle. Each is represented by a two-component object with components corresponding to the two spin projections. When the electron is represented by a local field, however, one needs four components because the two-component electron subspace has nonvanishing projections on all four components in configuration space. The nonvanishing of these projections results in turn from the original commutators for X^μ which came from requiring that P^μ generate translations in space-time (or equivalently that the local fields are found by Fourier transform). Our formalism at this point exactly parallels the standard Dirac theory.

The transformations between the various bases are most easily found as follows: The unitary transformation from γ_5 diagonal to γ^0 diagonal is well known (see the Appendix). The transformation from k^μ diagonal to y^μ diagonal (simultaneous with either γ_5 or γ^0 diagonal) is simply $(2\pi)^{-2}\exp(ik_\mu y^\mu)$, giving the four-dimensional Fourier transform. We thus need to connect only one of these bases to the one with $\gamma\cdot P$ diagonal. One first performs the unitary transformation from $|k^\mu,\gamma^0,\sigma\rangle$ to a helicity representation and then performs the unitary transformation to diagonalize $\gamma\cdot P$. By combining these transformations one may go

[4] S. Weinberg, Phys. Rev. 133, B1318 (1963).

from any one basis to any other. The important point is that the Dirac equation is "solved" by the transformation, and the components of the transformation constitute the "solution," as can be seen by inspection in the Appendix. This transformation can be used to find X^μ, γ^μ, $M^{\mu\nu}$, etc., in the Poincaré representations. In the Appendix it is shown that the transformation to the $\gamma\cdot P>0$ part of the Poincaré basis form, for example, the γ^0-diagonal basis is precisely the Foldy-Wouthuysen[5] transformation. Furthermore, it can be seen from the fact that the metric,

$$\begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix},$$

is invariant under the transformation that such a transformation can be cast into the form of a Lorentz transformation on the spinor variables.[6] More precisely, since the transformation is functionally dependent upon the momentum operators, we find that the transformation when written in the form of a Lorentz transformation depends upon a velocity which is written as a function of momentum operators. Thus it transforms with different "velocities" when acting on states with different momenta. Furthermore, the transformation commutes with P^μ and thus does not alter the four-momenta of a state, but acts only on the spinor variables of the space. The position operator X^μ in the Poincaré basis thus becomes a direct four-vector generalization of the Foldy-Wouthuysen position operator because the three-vector parts are identical. But we emphasize that in performing this transformation we are looking at the same algebra of operators, only in a different representation. Apparent differences arise from the fact that they act on particles which are irreducible Poincaré representations.

The metric $\gamma^0\epsilon(P^0)$ retains its diagonal form in the Poincaré basis because $U^+\gamma^0\epsilon(P^0)U=\gamma^0\epsilon(P^0)$, where U represents the unitary transformation $\langle k,\epsilon(P^0),\epsilon(\gamma\cdot P),\mathbf{k},w^0|k,\epsilon(P^0),\gamma^0,\mathbf{k},\sigma\rangle$ taking one from the original (γ^0) basis to the Poincaré basis. Thus the metric is diagonal on the representation space $|k,\epsilon(P^0),\epsilon(\gamma\cdot P),\mathbf{k},w^0\rangle$ having the form

$$\begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix} \text{ on } \begin{pmatrix}|\gamma\cdot p>0\rangle\\ |\gamma\cdot p<0\rangle\end{pmatrix}.$$

The scalar product is thus

$$\langle k',\epsilon(k^{0\prime}),\epsilon(\gamma\cdot k'),\mathbf{k}',w^{0\prime}|\eta|k,\epsilon(k^0),\epsilon(\gamma\cdot k),\mathbf{k},w^0\rangle$$
$$=\epsilon(\gamma\cdot k)\frac{|k^0|}{k}\delta(k'-k)\delta^3(\mathbf{k}'-\mathbf{k})$$
$$\times\delta_{\epsilon(k^{0\prime}),\epsilon(k^0)}\delta_{w^{0\prime},w^0}\delta_{\epsilon(\gamma\cdot k'),\epsilon(\gamma\cdot k)}$$

[where it is to be noted that the metric $\gamma^0\epsilon(k^0)=\epsilon(\gamma\cdot k)$ for $m>0$]. The completeness relation may be written as

$$1=\int kdk\frac{d^3\mathbf{k}}{|k^0|}\sum_{\epsilon(k^0),\epsilon(w^0),\epsilon(\gamma\cdot k)}|k,\epsilon(k^0),\epsilon(\gamma\cdot k),\mathbf{k},w^0\rangle$$
$$\times\langle k,\epsilon(k^0),\epsilon(\gamma\cdot k),\mathbf{k},w^0|,$$

which also acts as a projection operator onto the physical states since imaginary and negative mass is not included in the integration. The totally invariant scalar product for a field $|\psi\rangle$ is given by

$$\langle\psi|\eta|\psi\rangle=\langle\psi|\eta 1|\psi\rangle$$
$$=\sum_\alpha[\psi_\alpha^*(\gamma\cdot P>0)\psi_\alpha(\gamma\cdot P>0)$$
$$-\psi_\alpha^*(\gamma\cdot P<0)\psi_\alpha(\gamma\cdot P<0)],$$

where $\psi(\gamma\cdot P>0)$ is the projection of ψ onto the $\gamma\cdot P>0$ subspace and where α refers to the remaining indices. The various scalars $\gamma\cdot P$, $\epsilon(P^0)$, γ_5, and mixed products of these may be inserted in the invariant bilinear form to give various bilinear forms which are Poincaré-invariant bilinear forms and have obvious inversion properties under space and time inversion. We note that since the metric is definite within each Poincaré subspace, it follows that these irreducible Poincaré representations are unitary.

The nature of the $m=0$ states is not obvious using the Poincaré-invariant basis since $\epsilon(\gamma\cdot P)=0$, i.e., the sign multiplicity is lost. Furthermore, there appear to be eight $m=0$ states viewed in the representation $|k=0,\ \epsilon(k^0),\ \epsilon(b_1),\ \mathbf{k},\ w^0\rangle$. However, the transformation taking one from the γ_5 basis to the Poincaré basis results in states $|m=0\rangle$ which satisfy $\gamma\cdot P|m=0\rangle=0$. It is well known[7] that this equation reduces to $\epsilon(P^0)=i\gamma_5\epsilon(w^0)$ which admits only four states. The eigenvalues of $i\gamma_5=\epsilon(b_1)$ thus replace those of $\epsilon(\gamma\cdot P)$ but are constrained by the equation $i\gamma_5=\epsilon(P^0)\epsilon(w^0)$. Thus the labeling $|\mathbf{k},\epsilon(k^0),w^0\rangle$ for $m=0$ states is sufficient. The other four $m=0$ states appear to be connected to the limit $m\to 0$ from imaginary mass states. Each of the four $m=0$ states constitutes a Poincaré-invariant subspace. They are mapped into one another under the inversion group.

There is always some concern for the probability interpretation when the representation space has an indefinite bilinear form or scalar product.[8] This is because there are null vectors in the space which cause difficulty with normalization and the computation of expectation values. When the space can be separated into two subspaces, $|+\rangle$ and $|-\rangle$, such that the metric takes the form

$$\begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix},$$

[5] Leslie L. Foldy and Siegfried A. Wouthuysen, Phys. Rev. **78**, 29 (1950); Leslie L. Foldy, *ibid.* **102**, 568 (1956).

[6] This observation was first made by Herbert Jehle and William C. Park, Phys. Rev. **137**, B760 (1965).

[7] For example, Silvan S. Schweber, *Relativistic Quantum Field Theory* (Row, Peterson, Evanston, Ill., 1961), p. 111.

[8] K. L. Nagy, *State Vector Spaces with Indefinite Metric in Quantum Field Theory* (Noordhoff, Groningen, The Netherlands, 1966).

then the difficulties may be overcome by restricting the physical space to those vectors which lie exclusively in one or the other subspaces. In order to achieve an unambiguous theory, one must have a unique method of achieving the separation into subspaces. In the present formalism this unique separation is defined by the Poincaré basis; in particular, for unique spin the subspaces are eigenstates of $\gamma_{\mu_1\mu_2\cdots\mu_{2s}}P^{\mu_1}P^{\mu_2}\cdots P^{\mu_s}$. Thus a probabilistic interpretation is possible only if one requires that the physical subspace contain no mixture of these two subspaces. In the case of a spin-$\frac{1}{2}$ particle, this implies that physical states cannot be a linear combination of the two Dirac particles. Thus the requirement of a probability interpretation within the present framework with amplitudes at a given instant of proper time requires a superselection rule between the $\gamma\cdot P>0$ and $\gamma\cdot P<0$ subspaces. This restriction does not hold for $m=0$ states because there is no proper time for these states and thus no restriction on the probability interpretation of the amplitudes at a given instant of τ. The Hamiltonian which specifies the dynamics must keep one within the physical subspace.

The discussion presented for spin $\frac{1}{2}$ can be trivially carried through for all unique spin representations. With the exception of spin 0, a doubling is obtained for all other representations. One can eliminate the extra particle-antiparticle pair in a Poincaré-invariant manner if desired by dealing with only one subspace. Incidently, the mixed spin representation for the four-vector electromagnetic field is self-conjugate and thus there is no doubling of states for this case.

In view of the preceding discussion, the program for studying any of the other representations is also apparent. One could use any of the *XPM* representations supporting the inversion group to define a field. This representation is then to be decomposed into Poincaré-invariant subspaces by going to a new representation utilizing Poincaré invariants. The enumeration of all Poincaré-invariant restrictions, $Q|\psi\rangle=0$ (whether the restrictions are "wave equations" or "subsidiary conditions"), is equivalent to a restriction to a certain Poincaré subspace. The other subspaces can either be ignored as discussed or, if useful, assigned to other fundamental particles. One can require that the projection of operators onto the subspace of interest constitutes the observables. Then any transformation on the orthogonal subspaces leave the observables unchanged.

The construction of all Poincaré invariants for nonunique spins involves forming all Poincaré scalars supported by the representation. This in turn requires a knowledge of all tensor forms $\Gamma^{\mu_1\mu_2\cdots\mu_n}$ which are supported by the $S^{\mu\nu}$ representation and which have well-defined transformation properties under the Lorentz group (or equivalently have tensor commutation rules with $S^{\mu\nu}$). These tensor forms are in turn formed from all bilinear forms constricted on the $S^{\mu\nu}$ representation space. The decomposition of such forms is well known from decomposing the cross products of $S^{\mu\nu}$ representations with their conjugate representations. Thus there are in general an infinite number of "wave equations" which could apply to a particle of spin S if no other physical restrictions are imposed. One simply chooses a representation containing spin S and requires that only one of the Poincaré-invariant subspaces in the physical space. These representations differ in their transformation properties under $S^{\mu\nu}$ and inversion transformations. The local fields also have different compositions. The connection between our kinematical representations and those of the standard approach is very close. Traditionally, one considers a homogeneous Lorentz ($S^{\mu\nu}$) representation as functions of k^μ for a fixed mass. This is equivalent to taking the direct product $|k^\mu\rangle\otimes|b,s,\sigma\rangle$ for a fixed mass. The imposition of "wave" or constraint equations is then equivalent to a restriction to a given subspace. This is precisely the momentum representation space of the *XPM* algebra. Traditionally one then finds the local fields by Fourier transform from the $|k^\mu\rangle\otimes|b,s,\sigma\rangle$ space. This is again exactly what we achieve as a result of the X^μ commutation relations.[9]

IV. APPLICATIONS TO LEPTONS

One has traditionally progressed in field theory using only the space $\gamma\cdot P>0$ for spin-$\frac{1}{2}$ particles. That such a procedure is possible derives from the Poincaré invariance and mutual orthogonality of the two subspaces. We could also proceed this way in a proper-time formalism. It is to be noted, however, that the $\gamma\cdot P<0$ space represents a true Dirac particle-antiparticle pair with the four-momentum reversed.[10] For any given spin-$\frac{1}{2}$ particle-antiparticle pair, one could not distinguish whether they belonged to the $\gamma\cdot P>0$ or $\gamma\cdot P<0$ subspaces. In fact, one could place the electron and positron in the $\gamma\cdot P>0$ space and any other spin-$\frac{1}{2}$ particle-antiparticle pair in the other space.[11] One could then use the projections of operators onto one or the other subspace to obtain the standard theory. Of course, nothing would be gained if the particles bore no intimate relation to each other.

The electron and positron appear to be point particles and very accurately described by the Dirac field. It is strongly suggested by the above work that one ask if there is another particle whose properties are so similar to the electron that it can usefully be placed in the

[9] These considerations give the connection between our framework and the standard theory, enabling one to exploit most of our considerations without using a proper-time dynamics.

[10] This has been realized for a long time; the difference arises here from the property that the representation space contains *both* $\gamma\cdot P>0$ and $\gamma\cdot P<0$ in a *single* representation.

[11] We emphasize that the two signs of $\gamma\cdot P$ *each* contain both positive and negative energies and that we are *not* referring to the well-known procedure of recasting the negative-energy $\gamma\cdot P>0$ solution into the form of a positive-energy $\gamma\cdot P<0$ solution.

other subspace.[12] The perfect symmetry between the two subspaces leads one to expect that any operator form acting on one subspace which is contracted with an outside field would have a similar projection onto the other subspace and that projection would be contracted to the same field. Thus one would seek a second particle which had a symmetrical, if not identical, interaction form to that of the electron. In particular, the electromagnetic interaction should be identical for the electron and the other particle because the electromagnetic current γ^μ has a totally symmetrical form on the two spaces. Furthermore, the weak leptonic current formed from the electron-positron field and its associated neutrino, $L_e^\lambda = \bar{\psi}_e \gamma^\lambda (1 - i\gamma_5)\psi_\nu$, suggests that a similar current might be important when taken between the two omitted neutrinos and the other subspace. These are both just suggestive arguments based upon the symmetrical nature of the current operators as projected upon the two subspaces. The muon and its associated neutrino satisfy all of these requirements to a remarkable degree.

Let us then investigate the possibility of placing all leptons and their antiparticles within this single irreducible improper spin-$\frac{1}{2}$ representation of the XPM algebra. One is reminded that the unique spin-$\frac{1}{2}$ XPM representation $|m,\mathbf{k},\epsilon(P^0),\epsilon(\gamma\cdot P)w^0\rangle$ contains in one irreducible XPM representation two $[\epsilon(P^0) = \pm 1]$ irreducible Poincaré representations for each value of $\gamma\cdot P = \epsilon(\gamma\cdot P)m$, where $\gamma\cdot P$ ranges from $-\infty$ to $+\infty$. The four neutrinos can be placed in the $\gamma\cdot P = 0$ subspace, the electron and positron in the $\gamma\cdot P > 0$ subspace, and the muon and antimuon in the $\gamma\cdot P < 0$ subspace. We point out that the opposite choices for the massive particles would be equivalent.

The first question which arises concerns the mass difference between the electron and muon and the resulting asymmetry between the subspaces into which they are placed. From a kinematical point of view, one has a continuum of masses available from zero to infinity for either sign of $\gamma\cdot P$. Each mass, for each sign of $\gamma\cdot P$, forms a subspace for particle and antiparticle which is Poincaré invariant and inversion invariant. Therefore, these symmetry operators cannot take one between the two subspaces and consequently the masses are not required to be equal.[13] Thus the mass asymmetry in the $\gamma\cdot P$ spectrum: m_e, 0, $-m_\mu$, appears to be both allowable and unexplained. The actual physical masses must be introduced as knowledge about the incoming states and as allowable creation and annihilation operators in the interaction, thus giving only allowed transitions to the physically observed out-states. Perhaps some future dynamical theory will give these three values for the $\gamma\cdot P$ spectrum.

In order to develop a dynamical theory, we need to write a Poincaré-invariant leptonic Hamiltonian $H = H_0 + H_{\text{em}} + H_{\text{wk}}$. First consider H_0. In I we used a free-particle Hamiltonian $H_0 = +\sqrt{(P_\mu P^\mu)} = m$ with the idea that on spaces with indefinite metric, like spin $\frac{1}{2}$, we would reinterpret the creation and annihilation operators to make m positive definite in the same manner as the energy is ordinarily made positive definite. This procedure, although possible, does not lend itself as easily to the incorporation of electromagnetic interactions. An alternative procedure suggests itself in which this difficulty is not present and in which there is no "reinterpretation" of the operators. After looking at the various Poincaré-invariant bilinear forms, one notices that $\langle\psi|\eta\gamma\cdot P|\psi\rangle$ is positive definite since $\eta = \epsilon(\gamma\cdot P)$ on the Poincaré basis. Furthermore, it is inversion invariant and has the correct dimensions for a Hamiltonian. Since the physical space must be an eigenstate of $\gamma\cdot P$ (to avoid null vectors), it follows that $\gamma\cdot P$ will give the magnitude of the mass operator when acting on particle states. If we use $H_0 = \gamma\cdot P$ then the creation and annihilation operators need not be reinterpreted to make H_0 positive definite. The difficulties discussed in I with regard to this Hamiltonian do not appear on the Poincaré basis. This is related to the fact that the Poincaré basis is achieved by a Foldy-Wouthuysen type of transformation from the original basis. We emphasize though that the Dirac equation is a consequence of separating the space and holds for either Hamiltonian.

With regard to interactions, we see that a local field $|y\rangle$ projected onto the physical portion of the space is given by

$$|y\alpha\rangle = \sum_\beta \int_{\text{physical}} d^4k\, |k\beta\rangle\langle k\beta|\; |y\alpha\rangle$$

$$= \sum_\beta \int_{\text{physical}} d^4k \langle\beta|\alpha\rangle e^{ik_\mu y^\mu} |k\beta\rangle\,,$$

where α runs over the signs of γ^0 and σ, and β runs over the signs of $\gamma\cdot P$ and w^0. The matrix $\langle\beta|\alpha\rangle$ is the projection of the Poincaré basis onto the γ^0, σ basis and is equivalent to a solution of the Dirac equations (see the Appendix). More precisely, $\langle\beta|\alpha\rangle$ are the matrix elements of the generalized Foldy-Wouthuysen transformation. Thus, generally speaking, a "local field" is a linear combination of positive- and negative-frequency parts for $\gamma\cdot P = m_e$, 0, and $-m_\mu$. In practice one must take a small distribution of mass centered about these values and later let the width of the distributions approach zero to get mass eigenstates if desired. One can project out any particular subspace in forming the interactions. In a second quantized version, the above bra and ket vectors would be replaced by creation and annihilation operators. The integration

[12] Portions of this section were presented at the Southeastern Section Meeting of the American Physical Society, November 1969.

[13] This is in contrast to the fact that particles and antiparticles must have the same mass, because they are represented by positive- and negative-energy solutions connected by I_t and K_c, which both commute with mass.

over all space-time of a Hamiltonian density formed from these local fields then gives exact four-momentum conservation for the interaction. These fields are essentially the same as the standard local field in the limit of mass eigenstates. For example, the component of the local field projected onto the $\gamma\cdot P>0$ space is the same as the local Dirac field and satisfies the Dirac equation in the limit of a mass eigenstate.

We can construct the electromagnetic current in the following way. The projection operators for the electron and muon spaces, respectively, are $1+\gamma\cdot P/m$ and $1-\gamma\cdot P/m$. The electromagnetic current must be constructed from γ^μ as in the standard Dirac theory, but may be multiplied by any Poincaré scalar. The only two scalars which give the correct space-time inversion properties for the current are 1 and $\epsilon(\gamma\cdot P)$. The scalar $\epsilon(P^0)$ cannot be used because it would give the same charge for particle and antiparticle, i.e., j^0 would not change sign under charge conjugation. Thus the electromagnetic current is either of the form $\langle\psi|\eta\gamma^\mu|\psi\rangle$ or $\langle\psi|\eta\epsilon(\gamma\cdot P)\gamma^\mu|\psi\rangle$. If we require that the electromagnetic current vanishes for the neutrino, then only the latter form is permissible. Thus it follows that if the electron is assigned to the $P^0>0$, $\gamma\cdot P>0$ subspace then the positron is the $P^0<0$, $\gamma\cdot P>0$ subspace and the $\gamma\cdot P<0$ space must have $P^0>0$ for the μ^+ and $P^0<0$ for the μ^-. The result that the muons have the opposite sign of the energy for a given charge is compatible with the idea that the $\gamma\cdot P<0$ space is a Dirac particle with the four-momentum reversed. The local current density at a fixed instant of proper time may be found by inserting a complete set of local states as was done for spin 0 in I. Then H_{em} is formed as $\int d^4y\, j_\mu(y)A^\mu(y)$, giving the interaction at a fixed instant of proper time of the particle whose current is j_μ.

We would now like to find an expression for the weak leptonic current. By analogy to the standard expression, we can write L_e as

$$L_e^\lambda=\langle\psi_e|\eta\gamma^\lambda(1-i\gamma_5)|\psi_{m=0}\rangle$$
$$=\langle\psi_{m\neq 0}|\left(1+\frac{\gamma\cdot P}{m}\right)\eta\gamma^\lambda(1-i\gamma_5)|\psi_{m=0}\rangle$$

for the electron portion of the current. The states $|\psi\rangle$ used in this section should be thought of as creation operators for the respective states. The factor $1-i\gamma_5$ projects out the electron neutrino and antineutrino from the four possible $m=0$ states, while $1+\gamma\cdot P/m$ when acting on the $m\neq 0$ states will project out the electron subspace. However, we now find an unusual consequence of our previous result that the μ^- must be associated with the $P^0<0$ portion of the $\gamma\cdot P<0$ space. It is known that the muon interacts with the muon neutrino with the same helicity projections $1-i\gamma_5$, as the electron does with its neutrino. However, within our framework, to get the correct helicity projection for the μ^- it must be contracted with the $1+i\gamma_5$ portion of the $m=0$ space. Thus

$$L_\mu^\lambda=\langle\psi_\mu|\eta\gamma^\lambda(1+i\gamma_5)|\psi_{m=0}\rangle$$
$$=\langle\psi_{m\neq 0}|\left(1-\frac{\gamma\cdot P}{m}\right)\eta\gamma^\lambda(1+i\gamma_5)|\psi_{m=0}\rangle.$$

This may be seen directly by inspecting the solutions for $\gamma\cdot P<0$ in the Appendix and observing that in the γ_5 diagonal representation that the $P^0<0$, $\gamma\cdot P<0$ solution for the μ^- is identical to the $P^0>0$, $\gamma\cdot P>0$ solution for the e^- with its upper and lower components reversed. Consequently, the muon neutrino must be represented by the $1+i\gamma_5$ projection of the $m=0$ states in order to correspond to experiment. This gives us a way of distinguishing the muon and electron neutrinos by "space-time" variables as follows. The values of $\epsilon(P^0)$ and $\epsilon(w^0)$ are $+1-1$ for ν_e, -1, $+1$ for $\bar\nu_e$, -1, -1 for ν_μ, and $+1$, $+1$ for $\bar\nu_\mu$. One may now sum the two portions of the leptonic current to get $L^\lambda=\langle\psi_{m\neq 0}|[1-i(\gamma\cdot P/m)\gamma_5]\eta\gamma^\lambda|\psi_{m=0}\rangle$. H_{wk} is then formed as $H_{\text{wk}}=\int d^4y\, L_\lambda^\dagger(y)L^\lambda(y)$.

There is an intuitive way to see why the opposite $m=0$ solutions must be used for the muon neutrino without looking at the solutions. It is because the μ^- is associated with a negative-energy state. In fact, in the standard theory, if the e^- had been associated with the negative-energy Dirac solution one would have been forced to view neutrinos as the $1+i\gamma_5$ projections of $m=0$ particles. Thus the results which we have obtained can be stated within the standard theory if one associates the μ^- with a negative-energy state. The incorporation of both electron and muon neutrinos into one four-component massless field has been approached from several points of view in the literature.[14] What is new in our formulation is the distinction also of the electron and muon fields by a variable which arises naturally from a space-time framework [i.e., $\epsilon(\gamma\cdot P)$]. Furthermore, all lepton states are incorporated into one irreducible representation of a space-time algebraic structure proceeding from little more than the standard notions of locality.

V. GENERAL REMARKS

In conclusion we would like to consider some additional aspects of our formalism with particular emphasis on areas for further development. First of all, with this manifestly covariant approach to discrete operations as groups of isomorphisms on the *XPM* algebra, one treats X^0 and P^0 on an equal footing with the other operators. Since the interaction is carried by the invariant Hamiltonian H and not the energy P^0, it follows that time translation will commute with space inversion even for parity-violating interactions. It has been pointed out that this noncommutativity in the

[14] See, e.g., R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley-Interscience, New York, 1969).

standard theory is a source of ambiguity.[15] Thus, if a proper-time theory of weak interactions can be successfully implemented, we would expect this formal difficulty of the standard theory to be removed. It is also an advantage that the representations supporting discrete operations can be exhaustingly studied from a group-theoretical point of view. Secondly, it would be of interest to study in detail the Poincaré content of the mixed and infinite-dimensional representations as was done here for unique spin, in particular spin $\frac{1}{2}$.

One would like to set up a second quantized theory with a proper-time dynamics. Our work along these lines so far indicates that one may proceed in an almost identical fashion as with standard field theory by replacing time with proper time and the three-position by four-position. One begins by equating the anticommutation rules in momentum space between creation and annihilation fermion operators for the one-particle states to the invariant scalar product for one-particle states. The local field operators may be expressed in terms of the momentum-space operators, as in the Appendix, for one-particle states. Evaluation of the local commutators (at equal proper times) gives a generalization of the standard local Dirac commutator due to the inclusion of all physically acceptable masses in the local fields. The invariant Hamiltonian propagates each massive particle in terms of its own proper time.

H may be constructed as in Sec. IV for weak or electromagnetic interactions. Since there is a continuum of masses available mathematically, additional restrictions must be imposed to keep one within the framework of physically observable masses. This may be done by allowing in-states composed only of observed masses. and by restricting the local fields which form the interaction to be composed in momentum space of only allowable masses. This keeps the transitions to out-states physical. We are working on an attempt to prove that the electromagnetic and weak interactions in a proper-time framework gives the standard theory in the limit of mass eigenstates. But this problem is complicated by the fact that in the proper-time theory there is no operator $U(t_2,t_1)$ which takes the state from time t_1 to t_2. The proper-time dynamics gives an operator $U(\tau_2,\tau_1)$ which takes one from one proper time to another between wave packets spread out in space-time. It is only after computation of probabilities and an integration over proper time that one is able to relate the state on one spacelike surface to another. The hope would be that seeing even the same basic theory from an alternative point of view might suggest either an improvement in basic structure or in calculational procedures.

Finally, one can ask, to what extent it is useful to place all leptons in a single spin-$\frac{1}{2}$ representation. In Sec. IV we pointed out that this representation is quite compatible with the properties of the leptons and with the formal expressions for the weak and electromagnetic currents. Furthermore, the classification by the sign of $\gamma\cdot P$ gives the correct enumeration of types of leptons and the use of a single local field to incorporate all lepton states is formally appealing. This appeal would be even more satisfying if the program for a proper-time second-quantized theory can be implemented.

Another somewhat speculative point concerns the mass spectrum of the Hamiltonian $\gamma\cdot P$. Although none of the previous framework has hope of shedding light on this problem, one could attempt a somewhat phenomenological approach. It is well known that the leptons may be given hypercharge and isospin assignments and placed within an SU_3-triplet weight diagram. Although the meaning of such an assignment is not clear, the mass splittings can be written in terms of a phenomenological Hamiltonian by assuming certain transformation properties of the splittings. One can now ask if the labeling inherent in the "space-time" specification of the states [namely, $\gamma\cdot P$, w^0, and $\epsilon(P^0)$] can be used to specify a particle's location in such a weight diagram. If instead of I_3 and Y one considers the perpendicular axes of charge Q and U-spin component U_3, then the following identification can be made. The integrated zeroth component of the electromagnetic current can be identified as the charge $Q=\langle\psi|\eta\epsilon(\gamma\cdot P)\gamma^0|\psi\rangle$. U_3 may be identified with the zeroth component of another current, with $U_3=-\frac{1}{2}\langle\psi|\eta\epsilon(\gamma\cdot P)P^0|\psi\rangle$ possessing the values $-\frac{1}{2}$, 0, $+\frac{1}{2}$ for e^-, $\nu_{e,\mu}$, and μ^-, respectively. That Q and U_3 have the correct values for our previous lepton identification is easily checked in the rest frame of the particles. The charge axis must be shifted by an amount proportional to lepton number to give the correct weight values. However, the lepton number current appears to take a rather complicated form. If one first writes the mass splittings in terms of the internal variables and then converts via the above relations to "space-time" currents, it might be hoped that the resulting Hamiltonian would assume a form which would shed some light on these mass splittings. A difficulty is that the resulting form of the Hamiltonian is complicated by the structure of the leptonic current and so far the splitting which results is not transparent. We emphasize that the spirit of this conjecture is not necessarily related in any way to hadrons or strong-interaction physics. The idea is simply to extract information about the μ-e and e-ν mass splittings by analogy with the procedures used with hadron splittings.

APPENDIX

In this Appendix we first show the connection for spin $\frac{1}{2}$ between the abstract representation space and the standard forms for various operators like the Dirac matrices thereby establishing our notation. We then exhibit the transformations between the various bases. In terms of these transformations we can then exhibit

[15] This has been pointed out by T. D. Lee, in (e.g.) *Proceedings of The Second Hawaii Topical Conference in Particle Physics* (Hawaii U. P., Honolulu, 1967).

the solutions to eigenvalue equations which separate the space into irreducible Poincaré representations and then discuss the physical interpretation.

The invariants b_0 and b_1 are connected to operators on the spin-$\frac{1}{2}$ space by the equations

$$\tfrac{1}{2}S^{\mu\nu}S_{\mu\nu}=(\mathbf{S}^2-\mathbf{R}^2)=b_0^2+b_1^2-1=\tfrac{3}{2}I\,, \quad \text{(A1)}$$

$$\tfrac{1}{8}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma}=\mathbf{S}\cdot\mathbf{R}=-ib_0b_1=-\tfrac{3}{4}\gamma_5\,. \quad \text{(A2)}$$

The latter of these shows for spin $\frac{1}{2}$ ($b_0=\frac{1}{2}$, $b_1=\pm\frac{3}{2}$) that $\epsilon(b_1)=-i\gamma_5$. Thus the representation space $|k^\mu,\epsilon(b_1),\sigma\rangle$ is the γ_5-diagonal representation. The metric operator thus takes the form

$$\eta=\epsilon(P^0)\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

which is recognized to be $\epsilon(P^0)\gamma^0$ on the γ^5-diagonal basis. It is then straightforward to find γ by Lorentz transformation of γ^0:

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix},$$
$$\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3=i\epsilon(b_1)=i\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \quad \text{(A3)}$$

Certain general formulas which are useful are

$$[\gamma^\mu,\gamma^\nu]_+=2g^{\mu\nu}\,, \quad \text{(A4)}$$

$$S^{\mu\nu}=\tfrac{1}{4}i[\gamma^\mu,\gamma^\nu]\,, \quad \text{(A5)}$$

and

$$W_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}S^{\nu\rho}P^\sigma=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma\,, \quad \text{(A6)}$$

where

$$W^0=\mathbf{S}\cdot\mathbf{P} \quad \text{(A7)}$$

and

$$\mathbf{W}=\mathbf{S}P^0-\mathbf{P}\otimes\mathbf{R}\,. \quad \text{(A8)}$$

The matrix forms of $S_{\mu\nu}$ on the γ^5-diagonal representation are

$$S^1=\frac{1}{2}\begin{pmatrix}\sigma_1&0\\0&\sigma_1\end{pmatrix},\quad S^2=\frac{1}{2}\begin{pmatrix}\sigma_2&0\\0&\sigma_2\end{pmatrix},\quad S^3=\frac{1}{2}\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix}, \quad \text{(A9)}$$

$$R^1=-\tfrac{1}{2}i\begin{pmatrix}\sigma_1&0\\0&\sigma_1\end{pmatrix},\; R^2=-\tfrac{1}{2}i\begin{pmatrix}\sigma_2&0\\0&\sigma_2\end{pmatrix},\; R^3=-\tfrac{1}{2}i\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix},$$

and the action of the inversions on the spin portion of the representation

$$I_s=\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad I_t=\gamma^0\gamma^5=\begin{pmatrix}0&-i\\i&0\end{pmatrix},$$
$$I_{st}=\gamma^5=i\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad K_c=\gamma^2C_x, \quad \text{(A10)}$$

where C_x represents complex conjugation in the position basis.

The transformation to the γ^0 diagonal from the γ_5-diagonal representation is achieved by

$$U_{\gamma^5\to\gamma^0}(S^3)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}, \quad \text{(A11)}$$

giving (with spin S^3 diagonal in both representations)

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \gamma^i=\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix},\quad \gamma^5=-i\begin{pmatrix}0&1\\1&0\end{pmatrix}. \quad \text{(A12)}$$

The transformation from spin (S^3) diagonal to helicity (ω^0) diagonal is achieved by

$$U_{S^3\to\omega^0}(\gamma^0)=\begin{bmatrix}P^3+P&P^1+iP^2&0&0\\P^1-iP^2&-(P^3+P)&0&0\\0&0&P^3+P&P^1+iP^2\\0&0&P^1-iP^2&-(P^3+P)\end{bmatrix};$$

with the γ^0 diagonal before and after the transformation. The transformation which, with helicity diagonal, takes one from the γ^0 diagonal to the $\gamma\cdot P$ diagonal is

$$U_{\gamma^0\to\gamma\cdot P}(\omega^0)=\left(\frac{|P^0|+m}{2m}\right)^{1/2}\times\begin{bmatrix}1&\dfrac{\sigma\cdot\mathbf{P}}{P^0+m\epsilon(P^0)}\\\dfrac{\sigma\cdot P}{P^0+m\epsilon(P^0)}&1\end{bmatrix},$$

which results in upper and lower portions which are, respectively, positive and negative eigenstates of $\gamma\cdot P\epsilon(P^0)$. These transformations equivalently give the projections of any basis vector on any other basis vector and the composition of any basis vector in terms of other basis vectors. For example, the transformation $U_{S^3\to w^0}(\gamma^0)$ gives $|w^0+\rangle=\{(P^3+P)|S^3+\rangle+(P^1+iP^2)|S^3-\rangle\}$, where we have written the eigenvalue $\pm$ after the respective operator. From this equation it follows that

$$\langle S^3+|w^0+\rangle=\frac{P^3+|\mathbf{P}|}{[2|\mathbf{P}|(|\mathbf{P}|+P^3)]^{1/2}}$$

and thus the transformations U are composed of the projections of one unit basis vector onto another.

By combining these transformations one can write, for example, the positive-energy, positive-helicity solution to the Dirac equation ($\gamma\cdot P>0$) in momentum space, $|m,\mathbf{k},\gamma\cdot P+,\epsilon(P^0)+,w^0+\rangle$, in terms of another representation, say k^μ, γ_5, and S^3-diagonal basis. This

TABLE III. Poincaré subspaces expressed in the γ_5- and S^3-diagonal bases, with $\alpha\equiv|\mathbf{P}|/(|P^0|+m)$.

$\epsilon(\gamma\cdot P)$	+	+	+	+	−	−	−	−	0	0	0	0
$\epsilon(P^0)$	+	+	−	−	+	+	−	−	+	+	−	−
$\epsilon(w^0)$	+	−	+	−	+	−	+	−	+	−	+	−
Particle	e^-	e^-	e^+	e^+	μ^+	μ^+	μ^-	μ^-	$\bar{\nu}_\mu$	ν_e	$\bar{\nu}_e$	ν_μ
Solution	$\begin{pmatrix}1-\alpha\\0\\1+\alpha\\0\end{pmatrix}$	$\begin{pmatrix}0\\-1-\alpha\\0\\-1+\alpha\end{pmatrix}$	$\begin{pmatrix}-1-\alpha\\0\\1-\alpha\\0\end{pmatrix}$	$\begin{pmatrix}0\\1-\alpha\\0\\-1-\alpha\end{pmatrix}$	$\begin{pmatrix}1-\alpha\\0\\-1-\alpha\\0\end{pmatrix}$	$\begin{pmatrix}0\\-1-\alpha\\0\\1-\alpha\end{pmatrix}$	$\begin{pmatrix}1+\alpha\\0\\1-\alpha\\0\end{pmatrix}$	$\begin{pmatrix}0\\-1+\alpha\\0\\-1-\alpha\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

could be written as

$$|m,\mathbf{k},\gamma\cdot P+,\epsilon(P^0)+,w^0+\rangle$$
$$=\left[\frac{P^0+m}{8m|P|(|P|+P^3)}\right]^{1/2}\Bigg[(P^3+|P|)$$
$$\times\left(1-\frac{|\mathbf{P}|}{P^0+m}\right)|m,\mathbf{k},\gamma_5+,\epsilon(P^0)+,\sigma+\rangle$$
$$+(P^1+iP^2)\left(1-\frac{|\mathbf{P}|}{P^0+m}\right)|m,\mathbf{k},\gamma_5+,\epsilon(P^0)+,\sigma-\rangle$$
$$+(P^3+P)\left(1+\frac{|\mathbf{P}|}{P^0+m}\right)|m,\mathbf{k},\gamma_5-,\epsilon(P^0)+,\sigma+\rangle$$
$$+(P^1+iP^2)\left(1+\frac{|\mathbf{P}|}{P^0+m}\right)|m,\mathbf{k},\gamma_5-,\epsilon(P^0)+,\sigma-\rangle\Bigg].$$

Equivalently, if these coefficients are written as elements of a column vector, then they are immediately recognized as the positive-energy, positive-helicity solution to the Dirac equation using the basis γ^5, and S^3 diagonal. Thus the matrix elements of the transformation $U_{\gamma^0\to\gamma\cdot P}(w^0)U_{S^3\to w^0}(\gamma_0{}^0)U_{\gamma^5\to\gamma^0}(S^3)$ taking one from the γ^3, S^3 basis to the $\gamma\cdot P$, w^0 Poincaré basis constitutes, as column vectors, the solutions to $(\gamma\cdot P\pm m)|\psi\rangle=0$. It is also easily verified that the mass-zero solutions on the Poincaré basis satisfy $\gamma\cdot P|\psi\rangle=0$ and thus consist of four independent vectors.

Referring again to the transformation $U_{\gamma^0\to\gamma\cdot P}(\omega^0)$, one observes that half of this transformation is identical in form to the Foldy-Wouthuysen transformation. This can be seen by recalling that $U_{\gamma^0\to\gamma\cdot P}(\omega^0)$ results in upper and lower components which are positive and negative eigenstates, respectively, of $\gamma\cdot P\epsilon(P^0)$. Thus the Dirac solution $(\gamma\cdot P>0)$ with positive P^0 is the upper component and with negative P^0 is the lower component. By inserting these values for P^0 into $U_{\gamma^0\to\gamma\cdot P}(\omega^0)$, one gets the Foldy-Wouthuysen transformation apart from an over-all normalization $(|P^0|/m)^{1/2}$. That $U_{\gamma^0\to\gamma\cdot P}(\omega^0)$ is a more general transformation arises from the fact that the opposite sign choices for P^0 results in a transformation to the negative $\gamma\cdot P$ subspace with both positive and negative energies.

Since the form of the metric,

$$\eta=\gamma^0\epsilon(P^0)=\begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix},$$

is preserved by this transformation $U_{\gamma^0\to\gamma\cdot P}(\omega^0)$, it follows that it can be cast into the form of a Lorentz transformation on the two-dimensional space $|\gamma^0\pm\rangle$ (ignoring the helicity variable), i.e.,

$$\begin{pmatrix}\cosh\frac{1}{2}\theta & \sinh\frac{1}{2}\theta\\ \sinh\frac{1}{2}\theta & \cosh\frac{1}{2}\theta\end{pmatrix},$$

where $\tanh\theta=V/c=\mathbf{P}/P^0$.

We now write out explicitly in Table III the form of the Poincaré basis when expressed in terms of the γ^5- and S^3-diagonal basis for the simple case of $\mathbf{P}=(0,0,P)$. As discussed above, the $\gamma\cdot P>0$ portion constitutes the solution to the Dirac equation. Identifications of the leptons with this solution is shown along with the eigenvalues of the Poincaré covariants. For simplicity we define $\alpha\equiv|\mathbf{P}|/(|P^0|+m)$. As previously discussed, the form of the electromagnetic current suggests the identification of μ^- with the negative-energy, negative-$\gamma\cdot P$ subspace. In order to construct the correct helicity projections for muon-muon-neutrino leptonic currents, one is forced to use the opposite, $1+i\gamma_5$ neutrino projections for the muon neutrino as can be seen from Table III by inspection. This restriction then fixes the neutrino projections as shown.

Note added in proof. The group $\tilde{G}_5$ proposed by Aghassi *et al.*[2] is the group generated by the algebra obtained by adding a mass-squared operator S to our XPM algebra. As they show, the representations corresponding to different "internal" portions of the mass-squared operator are equivalent. Consequently, the representations which they obtain are the same as the infinite spin representations which we obtained in I. We agree with the criticism of Noga that a shortcoming of their framework is that they *only* have infinite spin representations. However, this inadequacy is a consequence of requiring that the homogeneous Lorentz representations be unitary and not merely a consequence of the commutation rule. We are considering both unitary and nonunitary homogeneous Lorentz representations, thus admitting unique, mixed, and infinite spin representations. The second criticism by Noga correctly points out that $\tilde{G}_5$ transforms physical mass states into unphysical (imaginary mass) states. These transformations (discussed in I) are generated by the covarient position operator X^μ. This problem may be avoided if one considers the XPM algebra as an algebra of observables rather than the group which it generates.